Section 10.1 - Sequences

A **sequence** is a list of numbers written in a certain order i.e)

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

or $\{a_1, a_2, \ldots, a_n, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples: $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

Ex) Find formula for $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, \frac{-6}{625}, \ldots\right\}$

$$\Rightarrow a_n = (-1)^n \frac{n+2}{5^n}$$

Ex) $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ $n \geq 3$ Fibonacci sequence

Definition: A sequence $\{a_n\}$ has a **limit**, L, written

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can ~~make~~ the a_n as close to L as we like for sufficiently large n.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges.

" " does not exist, we say sequence diverges

Definition: (Rigorous) A seq. has a **limit** L if for every $\varepsilon > 0$ there exists an integer N s.t. for $n \geq N$, $|a_n - L| < \varepsilon$

<u>Definition</u>: If $\lim_{n\to\infty} a_n = \infty \Rightarrow$ For every $M > 0$, there exists integer N s.t. for $n \geq N$, $a_n \geq M$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,
Sum rule, difference rule, products, quotients, powers
constants moved out, etc.

<u>Squeeze</u> <u>Thm</u>: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$
with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L$

If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

<u>Examples</u>: $a_n = \frac{n}{n+1}$, $a_n = \frac{\ln(n)}{n}$, $a_n = (-1)^n$, $a_n = \frac{(-1)^n}{n}$

$a_n = \frac{n!}{n^n} \Rightarrow$ Use partial sums and $a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$
$\Rightarrow 0 < a_n \leq \frac{1}{n}$ <u>Squeeze</u> <u>Thm</u>

<u>Defn</u>: The sequence $\{r^n\}$ converges for $-1 < r \leq 1$
and divergent otherwise

<u>Defn</u>: A seq. is <u>increasing</u> if $a_n < a_{n+1}$ $\forall n \geq 1$
" <u>decreasing</u> if $a_n > a_{n+1}$ $\forall n \geq 1$

It is <u>monotonic</u> if it is increasing <u>or</u> decreasing

The sequence $\left\{ \frac{3}{n+5} \right\}$ is decreasing.

Show $a_n = \frac{n}{n^2+1}$ is decreasing $n \geq 1$

WTS: $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$

Above iff $(n^2+1)(n+1) < n\left((n+1)^2+1\right)$

$\Leftrightarrow \quad n^3+n^2+n+1 < n^3+2n^2+2n$

$\Leftrightarrow \quad 1 < n^2+n$ True since $n \geq 1$

Defn: A sequence is bounded above if $\exists$ a number M s.t.

$a_n \leq M \quad \forall n \geq 1$

It is bounded below if $\exists$ a number m s.t.

$m \leq a_n \quad \forall n \geq 1$

If both hold, the sequence is bounded.

Thm: Every bounded monotonic sequence is convergent.

Example: Let $\{a_n\}$ be defined as $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n+6)$

$n = 1, 2, 3, \ldots$

Show increasing: Base case: $a_2 = 4 > a_1 = 2$

$a_{k+1} > a_k$ $\quad n=k$ step assumption

$a_{k+1}+6 > a_k+6$

$\frac{1}{2}(a_{k+1}+6) > \frac{1}{2}(a_k+6)$

$a_{k+2} > a_{k+1}$ ✓ $\Rightarrow$ increasing

Show $\{a_n\}$ is bounded

We will prove $\{a_n\} < 6 \ \forall n.$

Lower bound is trivial since $a_1 = 2 < a_2 < a_3 < \cdots$ since $\{a_n\}$ is increasing

$a_1 < 6$ trivially

Suppose true for $n = k$

$$\Rightarrow \quad a_k < 6$$

$$a_k + 6 < 12$$

$$\frac{1}{2}(a_k + 6) < \frac{1}{2} 12$$

$$\Rightarrow \quad a_{k+1} < 6 \ \checkmark$$

So by Thm, the seq. $\{a_n\}$ is convergent.

We do not know its limit though. The proof does not give us this. Call it L

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right)$$

$$= \frac{1}{2}(L + 6)$$

Since $a_n \to L$ $a_{n+1} \to L$ as well so the above can be done to see that

$$L = \frac{1}{2}(L + 6)$$

$$\frac{1}{2} L = 3$$

$$L = 6 \ \checkmark$$

$\blacksquare$