Section 10.2 - Series

When we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get the expression

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$$

Does this expression make sense?

Examples: $1+2+3+\dots+n+\dots$ Cumulative sum: $1, 3, 6, 10 \rightarrow \frac{n(n+1)}{2}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$

Cumulative sum/partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \dots, 1-\frac{1}{2^n}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Partial sums:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = \sum_{i=1}^{n} a_i$$

Definition: Given $\sum_{n=1}^{\infty} a_n$, $S_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum.

If $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists and a real number

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent i.e. $\sum_{n=1}^{\infty} a_n = S$.

Otherwise, the sum is divergent.

Thus we can say $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots \quad \text{is } \underline{\text{convergent}}$$

if $|r|<1$ and $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r|<1$

If $|r| \geq 1 \Rightarrow$ the series diverges.

<u>Example</u>: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$

$a = 5$ and $r = -\frac{2}{3} \Rightarrow |r| = \frac{2}{3} < 1$

$\Rightarrow$ convergent $= \dfrac{5}{1-(-\frac{2}{3})} = \dfrac{5}{\frac{5}{3}} = 3$

<u>Example</u>: Is $\displaystyle\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

$$= \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

$\Rightarrow$ divergent $|r| \geq 1$.

<u>Example</u>: Find sum of $\displaystyle\sum_{n=0}^{\infty} x^n$ for $|x|<1$

$\Rightarrow \quad a=1, r=x$ and $|x|<1$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

Example: Show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = 1$

Convergent

Show $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} = 2$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{3}{2} = 2.5$$

$$S_{16} > 1 + \frac{4}{2} = 3 \quad \Rightarrow S_{2^n} > 1 + \frac{n}{2}$$

$$\Rightarrow S_{2^n} \to \infty \text{ as } n \to \infty$$

$$\Rightarrow \{S_n\} \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Thm: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

The converse is not true in general.

Test for divergence: If $\lim_{n\to\infty} a_n$ does not exist or ~~converge~~

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} a_n = 0$ this tells us nothing!!!

Rules: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent then

① $\sum_{n=1}^{\infty} c\,a_n = c\sum_{n=1}^{\infty} a_n$ c a constant

② $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

~~③ $\sum_{n=1}^{\infty}$~~ And all the above are convergent

Example: Find $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{1}{2^n} \right)$

Note: A finite number of terms does not affect convergence.

Reindexing: Write $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ starting at

$n=0$

$n=5$

$n=-4$