Section 10.5 - Absolute Convergence, Ratio, Root Tests

Given a series $\sum_{n=1}^{\infty} a_n$, we can consider the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Since $\sum_{n=1}^{\infty} a_n$ may have positive and negative terms.

Defn: A series is called <u>absolutely</u> <u>convergent</u> if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

<u>Example:</u>

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{convergent } p\text{-series},$$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent

<u>Example:</u>

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges, harmonic series,}$$

not abs. conv,

We will see in 10.6 that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

<u>Defn:</u> A series $\sum a_n$ is <u>conditionally</u> <u>convergent</u> if it is convergent, but not absolutely convergent

<u>Example:</u> see last example.

Example: Determine whether $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is conv. or divergent

This ~~seq~~ series has positive and negative terms but does not alternate:

$$a_1, a_2, a_3, a_4$$
$$+, -, -, -, +, +, +, \ldots$$

We will apply Comparison Test with absolute value

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow \text{Convergent}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent

Theorem: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent

So the previous example is also Convergent.

Ratio Test: ① If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is abs. conv.

② If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges

③ If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ The Ratio Test is inconclusive, it tells us nothing!!

Example: Test $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

(it is absolutely convergent) Limit is $\frac{1}{3}$

Example: Test $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow \text{divergent by Ratio Test.}$$

Example: $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n! n!} = \frac{4 \cdot \cancel{4^n}(n+1)(n+1)\cancel{n!n!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{4^n n! n!}}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{4\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)} = \frac{2(n+1)}{2n+1} \longrightarrow 1 \text{ as } n \to \infty$$

Inconclusive!!

but... $\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1}$ $\quad a_{n+1} > a_n$ as numerator is always larger and $\frac{2n+2}{2n+1} > 1$

$\Rightarrow a_n \geq a_1 = 2$ and the n^{th} term does not approach 0 as $n \to \infty$, so by Divergence test $\Rightarrow$ sum is divergent.

Root Test

① If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent

② If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ is divergent

③ ~~If~~ $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \underline{1} \Rightarrow$ Inconclusive.

__Note!__ If either the Ration or Root test give $L=1$ don't try the other, as it will also be $\underline{1}$.

__Example:__ $\sum\limits_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n \Rightarrow \sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} \to \frac{2}{3} < 1$

as $n \to \infty$

$\Rightarrow$ absolutely convergent

__Example:__ $\sum\limits_{n=1}^{\infty} \frac{2^n}{n^3} \Rightarrow \sqrt[n]{|a_n|} = \frac{2}{(\sqrt[n]{n})^3} \to \frac{2}{1^3} > 1$

$\Rightarrow$ divergent

__Note:__ $\boxed{\text{Rearrangement}}$

__Theorem:__ If $\sum a_n$ is absolutely convergent with sum $\underline{S}$, then any rearrangement of $\sum a_n$ has the same sum

We __cannot__ rearrange a series <u>that is not absolutely convergent</u>