Section 10.7 - Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where the c_n are constant. A power series may converge some values of x but not others. A power series IS a function

$$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty} c_n x^n$$

More generally, we can have a power series centered elsewhere,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{centered at "a"}$$

Example: For what values of x does $\sum_{n=0}^{\infty} n! x^n$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} (n+1)|x| = \infty \qquad x \neq 0$$

by Ratio Test, diverges when $x \neq 0$, converges for $x = 0$

Example: For what values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \quad \text{as } n \to \infty$$

By Ratio Test, convergent for $|x-3| < 1$

divergent for $|x-3| > 1$,

The end pts must be tested by plugging in $x = 2, 4$

$$\Rightarrow [2, 4)$$

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are 3 possibilities

(i) Series only converges for $x=a$

(ii) series converges for all x

(iii) $\exists$ a positive number R s.t. the series converges if $|x-a| < R$ and diverges for $|x-a| > R$

The R is called the radius of convergence

case (i) $\Rightarrow R=0$, only one pt.

case (ii) $\Rightarrow R=\infty$, all points

Anything can happen at endpts since Ratio Test is inconclusive.

Some examples

Name	Series	Radius of Conv	Interval of Conv
Geometric Series	$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1,1)$
Example 1	$\sum_{n=0}^{\infty} n!\,x^n$	$R=0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R=1$	$[2,4)$
Example 3 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R=\infty$	$(-\infty,\infty)$

Other Examples to try

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\quad \left(\cdot\right) \left|-3x\sqrt{\frac{n+1}{n+2}}\right| = 3\sqrt{\frac{1+1/n}{1+2/n}}\,|x| \to 3|x|$

$|x| < \frac{1}{3}$, $R = \frac{1}{3}$

$\left(-\frac{1}{3}, \frac{1}{3}\right]$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ $\quad R=3$, $(-5,1)$