Section 10.8 - Taylor and Maclaurin Series

How can we find representations of functions we know as power series?

Suppose for any f, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If we let $x=a \Rightarrow f(a) = c_0$, so if we know f, then we know c_0. If we can take a derivative,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$\Rightarrow f'(a) = c_1$ if we know $f'(x)$ exists!

Differentiation and Integration of Power Series

Thm: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$ and f is differentiable (and conti.) on $(a-R, a+R)$ and

① $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

② $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

then the radii of convergence for ① and ② is the same as $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

$\blacksquare$

Ex) Express $\dfrac{1}{(1-x)^2}$ as a power series. State radius of convergence

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$ geometric series for $|x|<1 \Rightarrow R=1$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} n x^{n-1} \quad \text{or} \quad \sum_{n=1}^{\infty} (n+1) x^n}$$ $R=1$ by Thm

Ex) Express $\ln(1-x)$ as a power series. State R

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \int\left(\sum_{n=0}^{\infty} x^n\right)dx = \int \frac{1}{1-x}\,dx$$

So $$-\ln(1-x) = \int \frac{1}{1-x}\,dx = \int\left(\sum_{n=0}^{\infty} x^n\right)dx$$

$$= \sum_{n=0}^{\infty}\left(\int x^n\,dx\right)$$

$$\Rightarrow \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \qquad = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

and $R=1$ by Thm.

Ex) (a) Find power series for $\dfrac{1}{1+x^7}$

(b) Find $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series

Going back to the original problem,

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentiable

$\Rightarrow f(a) = c_0 \quad f''(a) = 2c_2$

$f'(a) = c_1 \quad f'''(a) = 2 \cdot 3 \cdot c_3$

⋮

In general $f^{(n)}(a) = n!\, c_n$

$$\Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Defn: As long as the above process is possible,

① $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, is the Taylor Series centered at a

② $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$ is the MacLaurin Series

We can do $f(x) = e^x$ as an example. (center at 0)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{d}{dx}e^x = e^x$$

Denote $T_n(x)$ as the nth degree Taylor Polynomial.

$T_1(x) = 1 + x$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

If $f(x)$ is the sum of its own Taylor series,

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Limit of partial sums

The remaining terms are denoted $R_n(x)$ = remainder

and $R_n(x) = f(x) - T_n(x)$

or

$$f(x) = T_n(x) + R_n(x)$$

and it must be true $\lim_{n\to\infty} R_n(x) = 0$

Thm: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le \delta$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le \delta$$

Bound for remainders

Ex) Using definition of Taylor series find Taylor series for

① e^x centered at $a=2$

② $\sin(x)$ centered at $a=0$ (Note: $|f^{(n+1)}(x)| \le 1$) $\forall x$

$$\Rightarrow |R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \quad \text{(Sq. Thm argument)}$$

$\rightarrow 0$ as $n \to \infty$

③ $\cos(x)$ centered at $a=0$ (diff series)

Ex) Evaluate $\int e^{-x^2}\,dx$ as an infinite series