# Section 10.10 - Binomial Series

The Taylor series generated by $(1+x)^m$ for $m$ constant is called the binomial series: You can use the power rule to see

$$f(x) = (1+x)^m \qquad f''(x) = (m)(m-1)(1+x)^{m-2}$$
$$f'(x) = (m)(1+x)^{m-1} \qquad f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$$

If $m$ is a positive integer, the series terminates after $(m+1)$ terms

If $m$ is not a positive integer or zero, it is an infinite series and converges for $|x|<1$

For $-1<x<1$, $(1+x)^m = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ for $0 \le n \le m$

Ex) $(1+x)^{\frac{1}{2}}$ $\left(1-\frac{1}{x}\right)^{\frac{1}{2}}$

$(1-x^2)^{\frac{1}{2}}$

## Evaluating Integrals

Ex) Express $\int \sin(x^2)\,dx$ as a power series

Ex) (a) Express $\int \frac{1}{1+x^2}dx$ as a power series

(b) Plug in $x=1$, what is the result telling us?

Ex) Evaluate $\lim_{x \to 0} \frac{\sin(x) - \tan(x)}{x^3}$ $\qquad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$= -\frac{1}{2}$ $\qquad \tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}$

Ex) In one throw of two dice, the probability of getting a roll of a 7 is $p = 1/6$. If you throw the dice repeatedly, prob. that a 7 appears for the first time on the $n^{th}$ roll is $q^{n-1}p$, where $q = 1-p = \frac{5}{6}$

Exp. # of rolls until first 7 is $\sum_{n=1}^{\infty} n q^{n-1} p$. Find the sum.

$E(X) = \sum x_i p_i$

$p$ until first success

$\Rightarrow q^{n-1} \cdot p$ failures, suc.

$\Rightarrow$ result

(i.e. coin flips)

Solution: Note that $S_n = \sum_{j=1}^{n} j \left(\frac{5}{6}\right)^{j-1}$

We know $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$

First, $(1-r)\sum_{j=1}^{n} ar^{j-1} = a + ar + ar^2 + \dots + ar^{n-1}$
$\qquad -ar - ar^2 - \dots - ar^{n-1} - ar^n$

$= a + ar^n$

$\Rightarrow \sum_{j=1}^{n} ar^{j-1} = \frac{a(1-r^n)}{1-r}$

$\Rightarrow \sum_{j=0}^{n-1} ar^j = \frac{a(1-r^n)}{1-r}$

$\Rightarrow \sum_{j=0}^{n} ar^j = \frac{a(1-r^{n+1})}{(1-r)}$

Assume $a = 1$

$\frac{d}{dr}\sum_{j=0}^{n} r^j = \frac{d}{dr}\left(\frac{(1-r^{n+1})}{(1-r)}\right)$ product rule

$\sum_{j=1}^{n} j r^{j-1} = \frac{1-r^{n+1}}{(1-r)^2} - \frac{(n+1)r^n}{1-r}$ ☆

$\Rightarrow \sum_{j=1}^{n} j\left(\frac{5}{6}\right)^{j-1} = \frac{1-(5/6)^{n+1}}{(1-(5/6))^2} - \frac{(n+1)(5/6)^n}{1-5/6}$

The limit of partial sums, is the sum

$\Rightarrow \lim_{n\to\infty} \sum_{j=1}^{n} j\left(\frac{5}{6}\right)^{j-1} = \sum_{j=1}^{\infty} j\left(\frac{5}{6}\right)^{j-1} = \lim_{n\to\infty} 36 - 36\left(\frac{5}{6}\right)^{n+1} - 6(n+1)\left(\frac{5}{6}\right)^n$

$= 36 - 0 - 0$

$= 36$

So $\sum_{n=1}^{\infty} n q^{n-1} p = \frac{1}{6}\sum_{n=1}^{\infty} n q^{n-1}$

$= \frac{1}{6} \cdot 36 = 6 \Rightarrow \boxed{E(\#\text{ rolls to first } 7) = 6}$