Section 11.2 - Calculus on Parametric Curves

Given $x = f(t)$, $y = g(t)$, how can we find $\frac{dy}{dx}$ as we would with regular functions?

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

Horizontal Tangent line when $\frac{dy}{dt} = 0$

Vertical Tangent line when $\frac{dx}{dt} = 0$ (if $\frac{dy}{dt} \neq 0$)

$$\Rightarrow \frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Ex) C is defined as $x = t^2$, $y = t^3 - 3t$

a) Show C has two tangents at $(3,0)$

b) Find points with horiz/vert tangent lines

c) Determine Concavity

d) Sketch

$t = \pm\sqrt{x}$

$\Rightarrow y = \sqrt{x}(x-3)$, $y = -\sqrt{x}(x-3)$ $\Rightarrow y^2 = x(x-3)^2$

a) $y = t^3 - 3t = t(t^2 - 3) = 0$

$t = 0,\ t = \pm\sqrt{3}$

$\Rightarrow (3,0)$ appears twice

$x = t^2 \Rightarrow x = 3$ twice

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

$$= \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3} \quad \Rightarrow \quad y = \sqrt{3}(x-3),\ y = -\sqrt{3}(x-3)$$

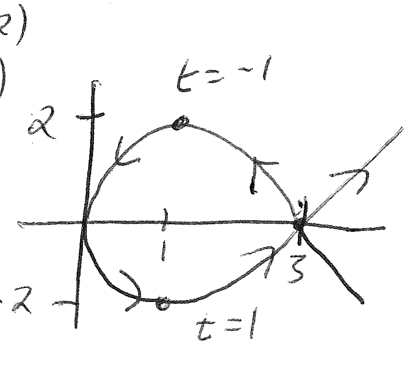
b) When $t = \pm 1 \Rightarrow$ horiz $\quad 3t^2 - 3 = 0 \quad (1,-2)$, $(1,2)$

$t = 0 \Rightarrow$ vert $\quad (0,0)$

$$\frac{d^2y}{dx^2} = \frac{3(t^2+1)}{4t^3}$$

C.C. up $t > 0$

C.C. down $t < 0$



Ex) Find tangent line to

(a) $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$ at the point where $\theta = \frac{\pi}{3}$

(b) Horiz/Vert tangent line pts.

(a) $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{\sqrt{3}/2}{1-1/2}$

$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$y = \frac{r}{2} = (1-\cos(\pi/3))r$

$$\boxed{y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)}$$

(b) Horiz $\Rightarrow$ $\sin\theta = 0$ and $\cos\theta \neq 1 \Rightarrow (2n-1)\pi \Rightarrow ((2n-1)\pi, 2r)$

vert $\Rightarrow$ $\cos\theta = 1$ $2n\pi$

but $\frac{0}{0} \Rightarrow$ limits

$\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim \frac{\cos\theta}{\sin\theta} = \infty$ $\Rightarrow$ vert tangents

$\theta \to 2\pi n^-$, $\frac{dy}{dx} \to -\infty$

Arc Length

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\cdot dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex) $x = \cos(t)$, $y = \sin(t)$

SA $S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

rotate $x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$ about x-axis to find $4\pi r^2$

Ex) $x = \cos(2t)$, $y = \sin(2t)$

Ex) $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$, $0 \le \theta \le 2\pi$

$8r$

$r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$

$= r\int_0^{2\pi} \sqrt{4\sin^2(\theta/2)}\, d\theta$

$= r\int_0^{2\pi} 2\sin(\theta/2)\, d\theta$