Section 6.4 - Surface Area of Revolution

The derivation for the surface area formulas is a mix between that of arc length and shells mixed together.

Definition 1 If $f(x) \geq 0$ is cont. differentiable on an interval $[a,b]$, the area of the surface generated by revolving $y=f(x)$ about the x-axis is

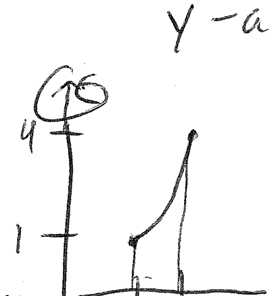
$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Definition 2 If $x=g(y) \geq 0$ is cont. differentiable on $[c,d]$, the area of the surface is given by

$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$

Examples

① Find the SA of $y=x^2$ from $(1,1)$ to $(2,4)$ rotated about the y-axis



$$SA = 2\pi \int_a^b x\sqrt{1+(f'(x))^2}\,dx$$

not correct

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}\,dx$$

$u = 1+4x^2$
$du = 8x\,dx$

$$= \frac{2\pi}{8}\int_5^{17} u^{1/2}\,du$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{17} = \boxed{\frac{\pi}{6}\left[17\sqrt{17}-5\sqrt{5}\right]}$$

② Find the surface area of the region generated by rotating $y=\sqrt{9-x^2}$ $\quad -2 \le x \le 2$ about the x-axis.

Solution:

$$S = 2\pi \int_{-2}^{2} f(x)\sqrt{1+(f'(x))^2}\,dx$$

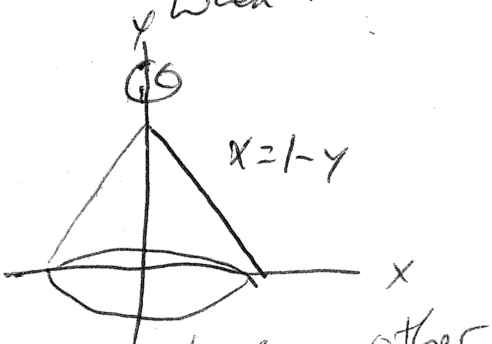
$$f'(x) = \frac{1}{2}(9-x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{9-x^2}}$$

$$\Rightarrow S = 2\pi \int_{-2}^{2} \sqrt{9-x^2}\cdot\sqrt{1+\frac{x^2}{9-x^2}}\,dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{9-x^2}\cdot\frac{3}{\sqrt{9-x^2}}\,dx$$

$$= 6\pi \int_{-2}^{2} dx = \boxed{24\pi}$$

③ Find the surface area of the region bounded by $x=1-y$, $0 \le y \le 1$ when revolved around the y-axis.



base $\quad$ other

$$A = \underbrace{\pi r\left(r + \sqrt{h^2+r^2}\right)}$$

$A =$

≈ 7.58

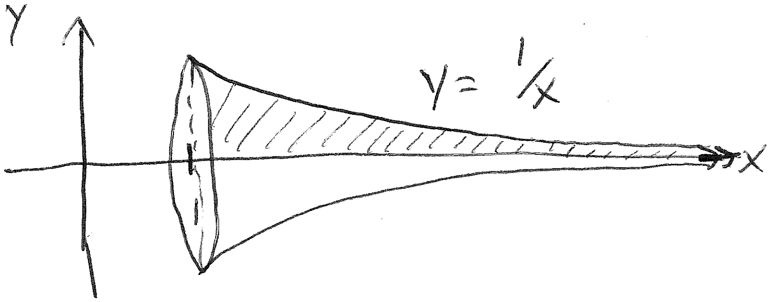
$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$

$$= \int_0^1 2\pi(1-y)\sqrt{1+(-1)^2}\,dy$$

$$= 2\pi\sqrt{2}\int_0^1 (1-y)\,dy$$

$$= \boxed{\sqrt{2}\,\pi}$$

(4) Let $D = \{(x,y) \mid x \geq 1, 0 \leq y \leq 1/x\}$ is rotated about the x-axis. Find its volume and show it's surface area is infinite.



"Gabriel's Horn"

Solution: Surface Area

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$$

We will be able to compute the integral exactly after Chapter 7. Notice that

$$\sqrt{x^4+1} > \sqrt{x^4} = x^2 \text{ for } x > 0.$$

$$\Rightarrow \quad 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3}\,dx = 2\pi \int_1^\infty \frac{1}{x}\,dx$$

$$= 2\pi \ln(x)\Big|_1^a$$

$$= 2\pi \cdot \lim_{a \to \infty} \ln(a)$$

$$= \boxed{\infty}$$

$\Rightarrow$ $\underline{S = \infty}$, or infinite surface area.

Volume (use disks)

$$V = \int_1^\infty \pi (R(x))^2\,dx = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \int_1^\infty \pi \frac{1}{x^2}\,dx$$

$$= \pi\left(-\frac{1}{x}\right)\Big|_1^a \Rightarrow = -\pi \lim_{a \to \infty} -\frac{1}{a} + \pi$$

$$= \boxed{\pi}$$