Section 7.5 – L'Hôpital's Rule + Indeterminate Forms

Suppose we want to know the behavior of $\frac{\sin(x)}{x}$ at $x=0$; or $\frac{\ln(x)}{x-1}$ at $x \to \infty$; We need limits.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0} \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln(x)}{x-1} = \frac{\infty}{\infty}$$

These expressions are indeterminate!! This means the could be equal to 0, ∞, or any finite number. More work is required to determine the correct value

Indeterminate Forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty\cdot 0, \infty-\infty, 0^0, 1^\infty, \infty^0$

L'Hôpital's Rule: Suppose f, g are diff. and $g'(x) \neq 0$ on (α,β) and $a \in (\alpha,\beta)$ except maybe at a

Suppose that

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
2. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ if limit on RHS exists or is $\pm\infty$.

Note: we can only use L'Hopital's for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ cases!! All others must be transformed into one of these two to use L'Hopital's Rule.

Examples $\qquad \frac{d}{dx}(\sec^2 x) = 2\sec^2(x)\tan(x)$

① $\lim\limits_{x\to 1} \frac{\ln x}{x-1}$

② $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$

③ $\lim\limits_{x\to\infty} \frac{\tan x - x}{x^3}$

④ $\lim\limits_{x\to 0^+} x\ln x$

⑤ $\lim\limits_{x\to(\pi/2)^-} \sec(x)-\tan(x)$

⑥ $\lim\limits_{x\to 0^+} (1+\sin(4x))^{\cot(x)}$

⑦ $\lim\limits_{x\to 0^+} x^x$

⑧ $\lim\limits_{x\to\infty} x^3 e^{-x^2}$

Solution: ⑥

$$\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)}$$

$$= \lim_{x\to 0^+} e^{\ln\left((1+\sin(4x))^{\cot(x)}\right)}$$

$$= \lim_{x\to 0^+} e^{\cot(x)\ln(1+\sin(4x))}$$

$$= e^{\lim\limits_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\tan(x)}}$$

$$= e^{\lim\limits_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin(4x)}}{\sec^2 x}}$$

$$= \boxed{e^4}$$

Mean Value Theorem: Let f, g be continuous on $[a,b]$ diff on (a,b), then $\exists\, c\in(a,b)$ $(g'(x)\neq 0)$

s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$