Section 7.7 - Hyperbolic Functions

Hyperbolic functions ~~s~~ simplify math expressions involving exponential functions.

Definitions:

$$\sinh(x) = \frac{e^{x}-e^{-x}}{2} \qquad \cosh(x) = \frac{e^{x}+e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

Identities:

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \sinh^2(x) = \frac{\cosh(2x)-1}{2}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\sinh^2(x) + \cosh^2(x) = \cosh(2x)$$

$$\tanh^2(x) = 1 - \operatorname{sech}^2(x)$$

Derivatives:

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

Integrals: Can be derived via the above formulas.

Ex) $\int \coth(5x)\,dx$

Ex) $\int \sinh^2(x)\,dx$

Ex) $\int \tanh\left(\frac{x}{7}\right)dx$

Identities for Inverse Hyperbolic Functions

→ see text Table 7.9 p. 427

Derivatives

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\operatorname{csch}^{-1}(x) = -\frac{1}{|x|\sqrt{1+x^2}} \quad (x\neq 0)$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \; (x>1) \qquad \frac{d}{dx}\operatorname{sech}^{-1}(x) = -\frac{1}{|x|\sqrt{1-x^2}} \quad (0<x<1)$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2} \quad |x|>1$$

$\hookrightarrow |x|<1$

The integral formulas can be formulated from above

Examples

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} \qquad \int \frac{dx}{x\sqrt{1+(\ln(x))^2}}$$

$$\int \frac{6\,dx}{\sqrt{1+9x^2}}$$

Application p 431 #77

A body of mass m falls from rest and encounters air resistance. The body's velocity follows the differential equation

$$m\frac{dv}{dt} = mg - kv^2 \qquad k, m, g \text{ are constant}$$

(a) Find solution to problem (ie. $v(t) = ?$)
(b) Limiting velocity (terminal speed)
(c) For 160lb person ($mg = 160$), t = seconds, $k = 0.005$, what is their limiting velocity

Solution: (a) $m\frac{dv}{dt} = mg - kv^2$

$$\Rightarrow \frac{dv}{dt} = g - \frac{k}{m}v^2$$

$$\Rightarrow \int \frac{1}{g - \frac{k}{m}v^2}\,dv = \int dt$$

$a^2 = g$

$u^2 = \frac{k}{m}v^2$

$u = \sqrt{\frac{k}{m}}\,v$

$du = \sqrt{\frac{k}{m}}\,dv$

$$\Rightarrow \frac{1}{\sqrt{g}}\sqrt{\frac{m}{k}}\tanh^{-1}\left(\frac{1}{\sqrt{g}}\sqrt{\frac{k}{m}}\,v\right) = t + C$$

$$\Rightarrow \sqrt{\frac{m}{gk}}\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = t + C$$

$$\Rightarrow \tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = \sqrt{\frac{gk}{m}}\,t + C$$

$$\Rightarrow \sqrt{\frac{k}{mg}}\,v = \tanh\left(\sqrt{\frac{gk}{m}}\,t + C\right)$$

$$\Rightarrow \boxed{v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)} \qquad C = 0$$

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right)$

$= \alpha \lim_{t\to\infty} \tanh(\beta t)$ $\left.\begin{array}{l} \alpha = \sqrt{\frac{mg}{k}} \\ \beta = \sqrt{\frac{gk}{m}} \end{array}\right\}$ constant

$= \alpha \lim_{t\to\infty} \frac{e^{\beta t} - e^{-\beta t}}{e^{\beta t} + e^{-\beta t}}$ multiply by $\frac{e^{\beta t}}{e^{\beta t}}$

$= \alpha \lim_{t\to\infty} \frac{e^{2\beta t} - 1}{e^{2\beta t} + 1}$

$\overset{L'H}{=} \alpha \lim_{t\to\infty} \frac{2\beta e^{2\beta t}}{2\beta e^{2\beta t}}$

$= \alpha \cdot 1 = \alpha = \boxed{\sqrt{\frac{mg}{k}}}$

(c) $mg = 160$ lbs

$k = 0.005 \frac{\text{lbs}}{\text{ft}}$ $\Rightarrow V_{terminal} = \sqrt{\frac{160}{0.005}}$

$g = 32 \frac{\text{ft}}{\text{s}^2}$

$= \sqrt{32000}$

$= 80\sqrt{5}$

$\approx \boxed{178 \text{ ft/s}}$

$u_t - \frac{1}{6} u_{xxt} + (\eta u)_x = 0$

$\eta_t - \frac{1}{6}\eta_{xxt} + \frac{1}{2}(u^2)_x = 0$

$+u_x$

$+\eta_x$