MATH 65B PRACTICE FINAL July 27, 2016

Name: KEY Score: ________ / 100

Student ID: ______________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										70
Score										
Pts. Possible	6	8	8	8	8	8	12	12	12	82

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours and 15 minutes to complete the exam.
- The test will be out of **70 points**. The highest possible score will be **82 points**. You can attempt as many of the questions as you wish, but keep in mind you are trying to get to the **70 points**.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- Higher point problems are harder, thus they are weighted more. In order to do well, you will have to attempt some of the more difficult problems.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions! Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for all $\lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$, for all $x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, for all $x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$, for all $x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$, for $x \in (-1,1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$, for $\lvert x\rvert \le 1$
$f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$, for $\lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$, for $\lvert x\rvert < 1$

1) (3 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{(2n-1)!}{(2n+1)!}$$

(3 pts.) (b) Determine whether the series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$$

(a) $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{\cancel{(2n-1)}\cancel{(2n-2)}\cancel{(2n-3)}\cdots\cancel{(2)}\cancel{(1)}}{(2n+1)(2n)\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{(2)}\cancel{(1)}}$

$a_n = \dfrac{1}{2n(2n+1)} \Rightarrow \lim\limits_{n\to\infty} \dfrac{1}{2n(2n+1)} = 0$

$\boxed{\text{convergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e\,\frac{e^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1} \left(= \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\right)$

$a = e \quad r = \dfrac{e}{3}$

$= \dfrac{e}{1-\frac{e}{3}}$

$\dfrac{e}{3} < 1 \Rightarrow$ convergent geometric series

$= \boxed{\dfrac{3e}{3-e}}$

2) (6 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n^2+4}$$

Integral Test: Consider $f(x) = \frac{1}{x^2+4} = (x^2+4)^{-1}$

① Contiunous: f defined for all x

② Positive: $f > 0$ for all x

③ Decreasing: $f'(x) = \frac{-2x}{(x^2+4)^2} < 0$ for $x > 0$

Integral test applies

$$\Rightarrow \int_2^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_2^{t}$$

$$= \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\tan^{-1}(1)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

$\Rightarrow$ convergent by integral test.

3) (8 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$$

Limit Comparison Test

$$b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \cdot \frac{n^{4/3}}{1} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{\sqrt[3]{n^7+n^2}} \cdot \frac{\frac{1}{n^{7/3}}}{\frac{1}{n^{7/3}}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n^5}}} = 1 < \infty$$

$\Rightarrow$ Since $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a convergent p-series

then $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^5}}$ is convergent by limit comparison test.

4) (4 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$$

(4 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

(a) Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+3)!} \cdot \frac{(n+2)!}{2^n n!} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{(n+2)!}}{(n+3)\cancel{(n+2)!}}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{n+1}{n+3}$$

$$= 2 > 1 \Rightarrow \boxed{\text{divergent}}$$

(b) Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{(n!)^n}{n^{4n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n!}{n^4} = \infty$$

$$\Rightarrow \boxed{\text{divergent}}$$

Neither can be conditionally convergent

5) (8 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^n}{n!}$$

We check absolute convergence first.

$$\sum_{n=2}^{\infty}\left|(-1)^n \frac{n^n}{n!}\right| = \sum_{n=2}^{\infty} \frac{n^n}{n!}$$

$$\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} \geq n \qquad \lim_{n\to\infty} n = \infty < \lim_{n\to\infty} \frac{n^n}{n!}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty \Rightarrow \text{divergent}$$

Same idea holds for checking with

$\sum_{n=2}^{\infty} (-1)^n \frac{n^n}{n!}$ which diverges by Test of Divergence.

6) (8 pts.) Find the radius of convergence and interval of convergence for the following power series:

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2(n+1)+1}\cdot\frac{2n+1}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}|x-3|\cdot\frac{2n+1}{2n+3}$$

$$= |x-3| < 1 \quad \Rightarrow \boxed{R=1}$$

$$-1 < x-3 < 1$$
$$+3 \quad +3 \quad +3$$

$$2 < x < 4 \quad \Rightarrow (2,4)$$

Check endpts

$$x=2 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{2n+1} = \sum_{n=0}^{\infty}\frac{1}{2n+1}$$ divergent, use comparison test

$$x=4 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \qquad \lim_{n\to\infty} b_n = 0 \qquad \frac{1}{2n+3} < \frac{1}{2n+1}$$

By Alternating Series, converges

$$\Rightarrow \boxed{(2,4]}$$

7) (6 pts.) (a) Compute the following integral using Taylor series.

$$\int \arctan(x^2)\,dx$$

(6 pts.) (b) Find the Taylor series centered at $a = \frac{\pi}{2}$ for

$$f(x) = \sin(x)$$

(a) $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $R=1$

$\arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

$\int \arctan(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}\,dx$

$= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{4n+2}}{2n+1}\,dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)}$ $R=1$

(b) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$

$R=\infty$

8) (6 pts.) (a) Eliminate the parameter in the for the following parametric equation

$$x = 9\cos(t) + 4 \quad y = 9\sin(t) + 1$$

(2 pts.) (b) Identify the type of graph from your result in part (a).

(4 pts.) (c) Compute $\frac{dy}{dx}$. What coordinates have a tangent line with slope 0? With undefined slope?

(a) $x = 9\cos(t) + 4 \qquad y = 9\sin(t) + 1$

$x - 4 = 9\cos(t) \qquad y - 1 = 9\sin(t)$

$\Rightarrow (x-4)^2 + (y-1)^2 = 81$

(b) Circle with center $(4, 1)$, radius 9

(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9\cos(t)}{-9\sin(t)} = \frac{\cos(t)}{\sin(t)}$

Horiz: $\frac{dy}{dx} = 0 \Rightarrow \cos(t) = 0$

$t = (n - \frac{1}{2})\pi$ (ie $\frac{\pi}{2}, \frac{3\pi}{2}$)

$\Rightarrow x = 4, y = 10, -8 \Rightarrow \boxed{(4,10) \; (4,-8)}$

Vertical: $\frac{dy}{dx} = \text{undef} \Rightarrow \sin(t) = 0$

$t = n\pi$ (ie: $0, \pi$)

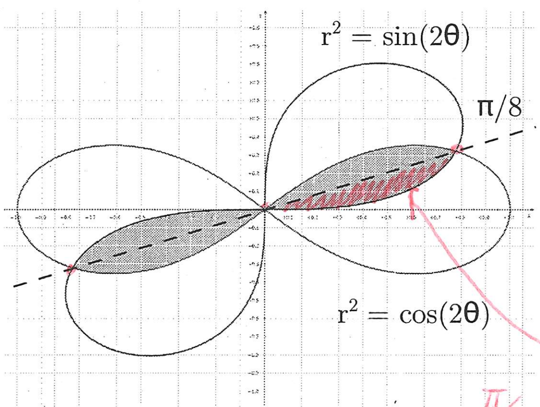
$\Rightarrow x = 13, -5 \quad y = 1 \Rightarrow \boxed{(13,1) \; (-5,1)}$

9) (12 pts.) Find the area of the region that lies inside the petals (the region is shaded in the labeled plot below).

Hint: Use the symmetry at $\frac{\pi}{8}$ to simplify the integral. How many pieces are there?

$$r^2 = \cos(2\theta)$$

$$r^2 = \sin(2\theta)$$



Find the intersection,

$\sin(2\theta) = \cos(2\theta)$

$\Rightarrow \dfrac{\sin(2\theta)}{\cos(2\theta)} = 1 \Rightarrow \tan(2\theta) = 1$

$\Rightarrow 2\theta = \dfrac{\pi}{4} \Rightarrow \theta = \pi/8$

By symmetry, $A_1 = \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta$ is the area of $\frac{1}{4}$ of the shaded region.

$$\Rightarrow A = 4A_1 = 4\int_0^{\pi/8} \frac{1}{2} \sin(2\theta)\, d\theta$$

$$= 2\int_0^{\pi/8} \sin(2\theta)\, d\theta$$

$$= -\cos(2\theta)\Big|_0^{\pi/8}$$

$$= \boxed{1 - \frac{\sqrt{2}}{2}}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST