MATH 65B PRACTICE MIDTERM June 30, 2016

Name: KEY Score: ________ / 50

Student ID: ______________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										50
Score										
Pts. Possible	X	X	X	X	X	X	X	X	X	54

INSTRUCTIONS FOR STUDENTS

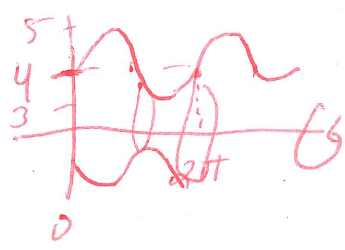
- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours and 15 minutes to complete the exam.
- The test will be out of **50 points**. The highest possible score will be **54 points**. You can attempt as many of the questions as you wish, but keep in mind you are trying to get to the **50 points**.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing.
- Higher point problems are harder, thus they are weighted more. In order to do well, you will have to attempt some of the more difficult problems.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit. Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Useful Formulas	Useful Formulas
$\dfrac{d(\arcsin(x))}{dx} = \dfrac{1}{\sqrt{1-x^2}} \quad \lvert u\rvert < 1$	$\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
$\dfrac{d(\arccos(x))}{dx} = -\dfrac{1}{\sqrt{1-x^2}} \quad \lvert u\rvert < 1$	$\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\dfrac{d(\arctan(x))}{dx} = \dfrac{1}{1+x^2}$	$\displaystyle\int \frac{dx}{u\sqrt{a^2-x^2}} = \frac{1}{a}\operatorname{arcsec}\left\lvert\frac{x}{a}\right\rvert + C$

1) Find the volume of the solid generated by revolving the function $y = f(x) = 4+\sin(x)$ with $0 \leq x \leq 2\pi$ about the x-axis.



$$V = \pi \int_a^b [R(x)]^2 \, dx$$

$$V = \pi \int_0^{2\pi} (4+\sin(x))^2 \, dx$$

Disk-Method

$$V = \pi \int_0^{2\pi} [16 + 8\sin(x) + \sin^2(x)] \, dx$$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$$V = \pi \int_0^{2\pi} \left(16 + 8\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$

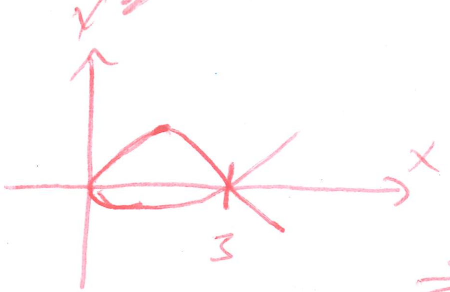
$$\Rightarrow V = \pi \left[16x - 8\cos(x) + \frac{1}{2}x - \frac{1}{4}\sin(2x)\right] \Big|_0^{2\pi}$$

$$V = \pi[32\pi - 8 + \pi - (-8)]$$

$$\boxed{V = 33\pi^2}$$

2) (a) Find the area of the surface generated by rotating the loop of the curve $9y^2 = x(3-x)^2$ about the x-axis.
(b) Find the area of the surface if the loop is rotated about the y-axis.

$y^2 = \frac{x(3-x)^2}{9}$



To find $\frac{dy}{dx}$, we need implicit differentiation $y = \frac{\sqrt{x}(3-x)}{3}$

$$\Rightarrow \frac{d}{dx}(9y^2) = \frac{d}{dx}\left(x(3-x)^2\right)$$

$$\Rightarrow 9(2y)\frac{dy}{dx} = x \cdot 2(3-x)(-1) + (3-x)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3-x)[-2x+3-x]}{18y} = \frac{(3-x)(-3x+3)}{18y}$$

$$\text{So } \left(\frac{dy}{dx}\right)^2 = \frac{(3-x)^2(3-3x)^2}{324y^2} = \frac{\cancel{(3-x)^2}(3-3x)^2}{324} \cdot \frac{9}{x\cancel{(3-x)^2}}$$

$$= \frac{(3-3x)^2}{36x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(3+3x)^2}{36x}$$

$$S = \int_0^3 2\pi y \, ds = 2\pi \int_0^3 \frac{\cancel{\sqrt{x}}(3-x)}{3} \cdot \frac{(3+3x)}{6\cancel{\sqrt{x}}} \, dx$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \frac{2\pi}{18}\int_0^3 (3-x)(3+3x)\,dx$$

$$= \frac{\pi}{9}\int_0^3 (9+6x-3x^2)\,dx$$

$$= \frac{9\pi}{3} = \boxed{3\pi}$$

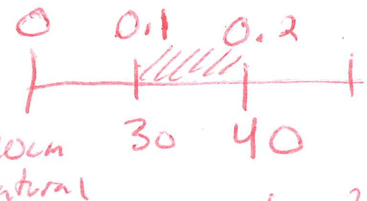
3) A spring has natural length of 20 cm. Compare the work, W_1, done by stretching the spring from 20 cm to 30 cm, to the work W_2, done by stretching the spring from 30 cm to 40 cm. How are W_1 and W_2 related?

Distance from 20cm to 30cm $\Rightarrow$ 0.1m

$$F(x) = kx \Rightarrow W_1 = \int_0^{0.1} kx\,dx = k\cdot\frac{1}{2}x^2\Big|_0^{0.1}$$

$$= \frac{1}{200}k$$

Distance from 30cm to 40cm is 0.1 Total, but is really 0.1 to 0.2 since



$$\Rightarrow W_2 = \int_{0.1}^{0.2} kx\,dx = k\frac{1}{2}x^2\Big|_{0.1}^{0.2} = \frac{3}{200}k$$

So $$\frac{W_2}{W_1} = \frac{\frac{3}{200}k}{\frac{1}{200}k} = 3 \Rightarrow \boxed{W_2 = 3W_1}$$

4) Solve the initial value problem

$$\frac{dy}{dx} = \frac{1}{x^2+1} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$$

$$\Rightarrow dy = \frac{1}{x^2+1}\,dx - \frac{2}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \int dy = \int \frac{1}{x^2+1}\,dx - 2\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow y(x) = \arctan(x) - 2\arcsin(x) + C$$

$$y(0) = \arctan(0) - 2\arcsin(0) + C$$

$$2 = 0 - 2(0) + C$$

$$\Rightarrow C = 2$$

$$\Rightarrow \boxed{y(x) = \arctan(x) - 2\arcsin(x) + 2}$$

5) Compute the following integral

$$\int x^2 \sin(2x)\,dx$$

$u = x^2 \qquad dv = \sin(2x)\,dx$

$du = 2x\,dx \qquad v = -\frac{1}{2}\cos(2x)$

$$\Rightarrow \int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) - \int \left(-\frac{1}{2}\cos(2x)\right)\cdot(2x)\,dx$$

$$= -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx \qquad \begin{array}{ll} u = x & dv = \cos(2x)dx \\ du = dx & v = \frac{1}{2}\sin(2x)dx \end{array}$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

$$= \boxed{\frac{1}{4}\left((1-2x^2)\cos(2x) + 2x\sin(2x)\right) + C}$$

7) Evaluate the integral

$$\int \frac{\sqrt{1+x^2}}{x}\,dx$$

Use $x = \tan\theta$, $a = 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \sec^2\theta \, d\theta$

$$\Rightarrow \int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} (1+\tan^2\theta)\, d\theta$$

$$= \int (\csc\theta + \sec\theta\tan\theta)\, d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C$$

$$= \boxed{\ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \sqrt{1+x^2} + C}$$

$$\int \csc x\,dx = \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x}\,dx$$

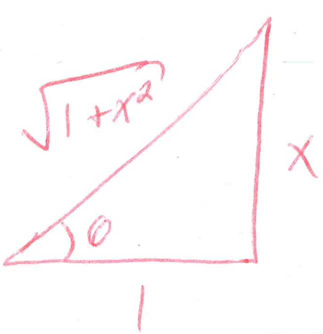
$\Rightarrow u = \csc x - \cot x$

$du = (\csc^2 x - \csc x \cot x)\,dx$

$$\Rightarrow = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|\csc x - \cot x| + C$$

$x = \tan\theta$



$$\csc\theta = \frac{\sqrt{1+x^2}}{x}$$

$$\cot\theta = \frac{1}{x}$$

$$\sec\theta = \frac{\sqrt{1+x^2}}{1}$$

6) Evaluate the integral

$$\int \sin^2(x)\cos^4(x)\,dx$$

$$= \int (1-\cos^2(x))\cos^4(x)\,dx$$

$$= \int \cos^4(x) - \cos^6(x)\,dx = \int (\cos^2(x))^2 - (\cos^2(x))^3\,dx$$

$$= \int \left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)^2 - \left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)^3 dx$$

$$= \int \left(\frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)\right) - \left(\frac{1}{8}+\frac{3}{8}\cos(2x)+\frac{3}{8}\cos^2(2x)+\frac{1}{8}\cos^3(2x)\right) dx$$

$$= \int \left(\frac{1}{8} - \frac{1}{8}\cos(2x) - \frac{1}{8}\cos^2(2x) + \frac{1}{8}\cos^3(2x)\right) dx$$

The only difficult integrals are

$$\int \cos^2(2x)\,dx = \frac{x}{2} + \frac{\sin(2x)}{4}$$

$$\int \cos^3(2x)\,dx = \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)$$

$$\Rightarrow \; = \frac{1}{8}x - \cancel{\frac{1}{16}\sin(2x)} - \frac{1}{16}x - \frac{1}{32}\sin(2x) + \cancel{\frac{1}{16}\sin(2x)} - \frac{1}{48}\sin^3(2x) + C$$

$$= \frac{1}{16}x - \frac{1}{32}\sin(2x) - \frac{1}{48}\sin^3(2x) + C$$

$$= \boxed{\frac{1}{16}\left[x - \frac{1}{2}\sin(2x) - \frac{1}{3}\sin^3(2x)\right] + C}$$

8) Evaluate the following integral.

$$\int \frac{1}{x^2-1}\,dx$$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx$$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$\Rightarrow 1 = A(x-1) + B(x+1)$$

$$\Rightarrow B = \tfrac{1}{2},\ A = -\tfrac{1}{2}$$

$$= \int \frac{-1/2}{x+1}\,dx + \int \frac{1/2}{x-1}\,dx$$

$$= \boxed{-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C}$$

9) Evaluate the integral

$$\int_{-1}^{0} \frac{e^{1/x}}{x^3}\, dx$$

$$= \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\, dx \qquad \text{Let } u = \frac{1}{x}$$

$$du = -\frac{1}{x^2}\, dx$$

$$= -\lim_{t\to 0^-} \int_{-1}^{1/t} u e^{u}\, du \qquad \text{Integration by Parts}$$

$$= -\lim_{t\to 0^-} e^{u}(u-1) \Big|_{-1}^{1/t}$$

$$= \lim_{t\to 0^-} \left(-2e^{-1} - \left(\frac{1}{t} - 1\right) e^{1/t}\right)$$

$$= -\frac{2}{e} - \lim_{s\to -\infty} (s-1)e^{s} \qquad \text{if you let } s = \frac{1}{t}$$

$$\Rightarrow t \to 0^{-}$$

$$\Rightarrow s \to -\infty$$

$$= -\frac{2}{e} - \lim_{s\to -\infty} \frac{s-1}{e^{-s}}$$

L'H

$$= -\frac{2}{e} - \lim_{s\to -\infty} \frac{1}{-e^{-s}}$$

$$= -\frac{2}{e} - 0 = \boxed{-\frac{2}{e}} \qquad \text{Convergent}$$

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END OF TEST