# The Multi-domain Hybrid Method for Calculating the Power Law Decay of Gravitational Waveforms with the Analytic Radiation Boundary Conditions

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# Introduction

- Schwarzschild black hole system
- Quasi-normal ringing
- Zerilli Equation
- Analytic Radiation Boundary Condition
- Spectral Methods
- Multidomain Spectral and Finite difference method
- Implementation
- Results
- Conclusion and Future Work

# Physical System

System

- Black hole at center of galaxy, point mass, free fall, gravitational waves

Schwarzschild Metric

- $$ds^2 = -\left(1 - \frac{2M}{r}\right) dt + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$
- M = mass of black hole

# Quasi-normal Ringing

- Black holes are always in a perturbed state
- Metric can be given by $g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$
- $g^0_{\mu\nu}$ is the spacetime of the non-pertubed black hole (ie. Schwarzschild)
- $\delta g_{\mu\nu}$ is the "small" perturbation such that $\delta g_{\mu\nu} << g^0_{\mu\nu}$
- Black hole emits gravitational waves
- Damped oscillations occur as a result
- This is called *quasi-normal ringing*
- For large time, $\tau$, this ringing is supressed by power-law late time tails
- Magnitude of the power is depeendent on a variety of factors
- arxiv - [Konoplya 2011]

# Zerilli Equation

Zerilli ([Zerilli 1970]) considered the previous system and derived the inhomogeneous Zerilli equation

$$\Psi_{tt} = \Psi_{r^* r^*} - V_l(r)\Psi,$$

$$r^* = r + 2M \ln\left(r/2M - 1\right)$$

The potential term in this equation is called the Zerilli potential

$$V_l(r) = \left(1 - \frac{2M}{r}\right) \frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3\left(\lambda r + 3M\right)^2},$$

With $\lambda = (l+2)(l-1)/2$, and $l$ refers to the orbital index, in our case for a Schwarzschild black hole, $l = 2$.

# Analytic Radiation Boundary Conditions

- Following the method in [Lau2005]
- We convert to dimensionless parameters
- $t = 2M\tau$ and $r = 2M\rho$ (Take $M = 1$).
- PDE becomes $\frac{\partial^2 \Psi}{\partial \tau^2} - \frac{\partial^2 \Psi}{\partial \rho_*^2} + V^Z(\rho)\Psi = 0$
- Potential becomes $V^Z(\rho) = \left(1 - \frac{1}{\rho}\right)\left(\frac{8n^2(n+1)\rho^3 + 12n^2\rho^3 + 18n\rho + 9}{\rho^3(2n\rho+3)^2}\right)$
- $n = \frac{1}{2}(l-2)(l+1)$, where $l$ is the orbital index, for us $l = 2$

# Laplace Transform Technique

- By taking the Laplace Transform of the wave equation
- $\frac{\partial^2 \hat{\Psi}}{\partial \rho_*^2} - (\sigma^2 + V^Z(\rho))\hat{\Psi} = 0$
- The value $\sigma$ denotes the frequency variable wrt. $\tau$
- This ODE is a special form of the confluent Heun Equation
- $\frac{d^2\Gamma}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{d\Gamma}{dz} + \frac{\alpha z - q}{z(z-1)}\Gamma = 0$
- We use the substitution $\hat{\Psi} = \rho\hat{\psi}$
- where $\hat{\psi} \approx \rho^{-1} e^{\sigma\rho} W_l(\sigma\rho)$
- $W_l(\sigma\rho) \approx \sum_{n=1}^{\infty} (\sigma\rho)^{-n}$

# Laplace Transform Technique (cont.)

- Differentiate wrt to $\rho$ and apply the inverse Laplace Transform
- $\frac{1}{U}\frac{\partial\Psi}{\partial\tau} + \frac{1}{T}\frac{\partial\Psi}{\partial\rho} + \frac{1}{T}\frac{\Psi}{\rho} = \rho^{-1}U(\rho)\psi_l(\tau,\rho) * \mathcal{L}^{-1}\left(\sigma\rho\frac{W_l'(\sigma\rho;\sigma)}{W_l(\sigma\rho;\sigma)}\right)$
- Where $T = F^{-1/2}(\rho)$, $U = F^{1/2}(\rho)$
- $F(\rho) = 1 - \rho^{-1}$, from the metric
- Evaluating at the boundary, which we call $\rho = \rho_B$
- $\left(\frac{\partial\Psi}{\partial\tau} + \frac{\partial\Psi}{\partial\rho_*}\right)\Bigg|_{\rho=\rho_B} = \frac{1-\rho_B^{-1}}{\rho_B}\int_0^\tau \omega_2(\tau-\tau';\rho_B)\Psi_2(\tau';\rho_B)d\tau'$
- With $\omega$ being the compressed Zerilli kernel and $\Psi$ being the history of the gravitational wave.

# Spectral Method Overview

For example, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$u(0,t) = u(l,t) = 0, \tag{2}$$

$$u(x,0) = f(x); \frac{\partial u}{\partial t}(x,0) = g(x). \tag{3}$$

- We can easily solve this PDE using separation of variables
- The time dependent ODE yields the solution

$$\frac{d^2 h_m(t)}{dt^2} = -\lambda h_m(t). \tag{4}$$

# Spectral Method Overview

- Let $h_m^n = h_m(t_n)$, then apply time stepping
- Using finite difference in the time variable:

$$\frac{h_m^{n+1} - 2h_m^n + h_m^{n-1}}{(\Delta t)^2} = -\lambda h_m^n, \tag{1}$$

- Solving for the $h_m^{n+1}$ term,

$$h_m^{n+1} = (-\lambda(\Delta t)^2 + 2)h_m^n - h_m^{n-1}. \tag{2}$$

- Numerical solution is found by using initial amplitude from the initial conditions and running the PDE through the time stepping process.
- In the spatial dimension, $x_j$ are chosen to be Chebyshev nodes and are evaluated on some grid values.

# Multidomain Spectral and finite difference method

- The multidomain method we use was developed in [Chakraborty,Jung 2011].
- Multi-domain spectral and finite difference methods is used for the spatial derivaties.
- Second order finite difference method based on the two step method is used for the temporal derivative.
- Computational domain is partitioned into the multi-domain spectral domains and the finite difference domain.
- Spectral domain uses the Chebyshev spectral method and the size of all the spectral domains is same.
- Our PDE is solved in each subdomain and the solution is patched across the domain interface.

# Patching for spectral domains and filtering

Patching

- We have two different types of patching depending on the interface:
- First case - adjacent spectral domains
- Second case - spectral and finite difference domains
- Solution is updated at the interfaces
- Details in [Chakraborty,Jung 2011]

Filtering

- Filters can be used to reduce non-physical reflections at the boundaries.
- In spectral domains, the filter of order $q$ is defined to be

$$\sigma(\theta) = e^{-\alpha|\theta|^q}, \quad \alpha > 0, \tag{3}$$

# Multidomain Spectral and finite difference method (cont.)

- In [Chakraborty,Jung 2011], the right boundary was placed at $\rho = 387.5$
- The theoretical power-law decay rate is $t^{-2l-3}$ where $l$ is the orbital index. Recall for our problem, that $l = 2$ is taken. So our goal is to get a power-law close to $t^{-7}$.
- In [Chakraborty,Jung 2011], $t^{-6.7}$ was obtained, which agrees well with the theoretical result.
- Our goal is to obtain the same accuracy or improve accuracy and lower computational time involved.
- Major difference is utilizing Lau's exact boundary condition.

# Boundary Conditions at $\rho_* = \rho_*^L$

- Outflow boundary condition at the left domain boundary is given by

$$\left( \frac{\partial \Psi}{\partial \tau} - \frac{\partial \Psi}{\partial \rho_*} \right) \bigg|_{\rho_* = \rho_*^L} = 0. \tag{4}$$

- Boundary condition is not exact due to potental function
- To remedy this, $\rho_*^L$ is chosen large enough so that unphysical reflections do not reach the measuring point of the waveform.
- The boundary condition is approximated by the first order finite difference method such as

$$\Psi(\tau + \Delta t) = \Psi(\tau) + \frac{\Delta t}{\Delta \rho_*} \left( \Psi(\rho_*^L + \Delta \rho_*) - \Psi(\rho_*^L) \right),$$

where $\Delta t$ and $\Delta \rho_*$ are the temporal and spatial grid spacing.

# Boundary Conditions at $\rho_* = \rho_*^R$

- Outflow boundary condition at the right domain boundary is given by Lau's NRBC
- For the implementation of the NRBC, we estimate the kernel $\omega_2$ using a power series and the compressed kernel values from [Lau2005] with $d = 10$, given by

$$\omega_2(\tau - \tau') \approx \sum_{k=1}^{d} \gamma_k e^{\beta_k(\tau-\tau')}. \quad (5)$$

- The values of $\beta_k$ and $\gamma_k$ are adopted from Table II for the compressed Zerilli kernels in [Lau 2005]. Those values in Table II in [Lau2005] were obtained specifically for those when $\rho_B = 15$.

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- Overview of the derivation in [Buli,Jung,Chakraborty2014]
- Start with the exact boundary condition from before

$$\left(\frac{\partial\Psi}{\partial\tau} + \frac{\partial\Psi}{\partial\rho_*}\right)\Bigg|_{\rho=\rho_B} = \frac{1-\rho_B^{-1}}{\rho_B}\int_0^\tau \omega_2(\tau-\tau';\rho_B)\Psi_2(\tau';\rho_B)d\tau' \quad (6)$$

- Taking the RHS of the equation, we simplyfy the expression

$$A(\tau+\Delta t) = \frac{F(\rho_B)}{\rho_B}\sum_{k=1}^{d}\int_0^{\tau+\Delta t}\gamma_k e^{\beta_k(\tau+\Delta t-\tau')}\Psi(\tau')d\tau', \quad (7)$$

- where $F(\rho_B) = 1-\rho_B^{-1}$.

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- Rewritting the integral on the right hand side, we get

$$C_k(\tau + \Delta t) = \sum_{k=1}^{d} \gamma_k e^{\beta_k(\tau+\Delta t)} \int_0^{\tau+\Delta t} e^{-\beta_k \tau')} \Psi(\tau') d\tau'. \qquad (8)$$

- This integral can be broken into two parts; the first integral being evaluated from 0 to $\tau$ and the second being evaluated from $\tau$ to $\tau + \Delta t$.

$$C_k(\tau + \Delta t) = e^{\beta_k \Delta t}(C_k(\tau) + B_k(\tau)), \qquad (9)$$

where $C_k(\tau)$ and $B_k$ are

$$C_k(\tau) = \gamma_k e^{\beta_k \tau} \int_0^{\tau} e^{-\beta_k \tau'} \Psi(\tau') d\tau', \qquad (10)$$

$$B_k(\tau) = \gamma_k e^{\beta_k \tau} \int_{\tau}^{\tau+\Delta t} e^{-\beta_k \tau')} \Psi(\tau') d\tau'. \qquad (11)$$

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- We can then write $A(\tau + \Delta t)$ in the compressed form

$$A(\tau + \Delta t) = \frac{F(\rho_B)}{\rho_B} \left[ \sum_{k=1}^{d} e^{\beta_k \Delta t} C_k(\tau) + \sum_{k=1}^{d} e^{\beta_k \Delta t} B_k(\tau) \right]. \qquad (12)$$

- For the integral of $B_k(\tau)$, we use a first order approximation of $\Psi$ to get

$$\Psi(\tau') = \frac{\Psi(\tau + \Delta t) - \Psi(\tau)}{\Delta t} (\tau' - \tau) + \Psi(\tau), \qquad (13)$$

- After some manipulation, we get $B_k(\tau)$ is

$$B_k = \gamma_k \left( \left[ \frac{\Psi(\tau + \Delta t) - \Psi(\tau)}{\Delta t} \right] D_k + \Psi(\tau) E_k \right), \qquad (14)$$

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- Where $D_k$ and $E_k$ are coefficients defined as

$$D_k = \frac{1}{\beta_k^2}(1 - e^{-\beta_k \Delta t}[1 + \beta_k]), \tag{15}$$

$$E_k = \frac{1}{\beta_k}[1 - e^{-\beta_k \Delta t}]. \tag{16}$$

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- To find the boundary value of $\Psi(\rho_*^R)$, we use the explicit forward difference method using the exact NRBC which is given by

$$\frac{\Psi(\rho_*^R, \tau + \Delta t) - \Psi(\rho_*^R, \tau)}{\Delta t} = -\frac{\Psi(\rho_*^R, \tau) - \Psi(\rho_*^R - \Delta\rho_*, \tau)}{\Delta\rho_*} + A(\tau + \Delta t) \tag{17}$$

- where $\Psi(\rho_*^R, \tau + \Delta t)$ is the unknown value of $\Psi$, $\Psi(\rho_*^R, \tau)$ and $\Psi(\rho_*^R - \Delta\rho_*, \tau)$ are the previous values of $\Psi$, and $\Delta\rho_*$ is the grid spacing at the domain boundary. With $A(\tau + \Delta t)$, $\Psi(\rho_*^R, \tau + \Delta t)$ is given by

$$\Psi(\rho_*^R, \tau + \Delta t) = \frac{\Psi(\rho_*^R, \tau) - H(\Psi^n) + \Delta t \frac{F(\rho_*^R)}{\rho_*^R}[J_k(\tau)]\Psi(\rho_*^R, \tau)}{1 - \frac{F(\rho_B)}{\rho_B}\sum_{k=1}^{d} e^{\beta_k \Delta t}\gamma_k D_k}, \tag{18}$$

# Boundary Conditions at $\rho_* = \rho_*^R$ (cont.)

- where $H(\Psi^n)$ and $J_k(\tau)$ are defined as

$$H = \frac{\Delta t}{\Delta \rho_*}[\Psi(\rho_*^R, \tau) - \Psi(\rho_*^R - \Delta\rho_*, \tau)], \tag{19}$$

$$J_k(\tau) = \left[\sum_{k=1}^{d} e^{\beta_k \Delta t}(C_k(\tau) + \gamma_k \left(E_k - \frac{1}{\Delta t} D_k\right)\right]. \tag{20}$$

- Finally, we have the expression that we can use in the numerical method!

# Results

- We solve our original PDE using the second order two step method in the temporal coordinate and the hybrid method for the spatial derivatives.
- We take the smooth Gaussian profile to be our initial condition

$$\Psi(\rho_*, \tau = 0) = \exp(-\frac{1}{10}(\rho_* - \rho_*^o)^2),$$

- With $\rho_*^o = -50$. The polynomial order, $p_s$, and the total number of subdomains, $N_t$, are chosen such that the Gaussian profile is smooth enough within each subdomain. We choose $p_s = 16$ and $N_t = 1000$.

# Results

- The number of grid points in the finite difference domain is chosen such that the grid homogeneity is maintained across the domain interface between the spectral and finite difference domains. The initial Gaussian profile satisfies the following condition at $\tau = 0$:
$$\frac{\partial \Psi}{\partial \tau} + \frac{\partial \Psi}{\partial \rho_*} = 0. \tag{21}$$
- The interface of the spectral and finite difference domains is at $\rho = 12$. The outer boundary of the finite difference domain is at $\rho = \rho_B = 15$.

# Results: Case 1

- Boundary condition at $\rho_B$ is $\frac{\partial \Psi}{\partial \tau} + \frac{\partial \Psi}{\partial \rho_*} = 0$ for all $\tau > 0$.

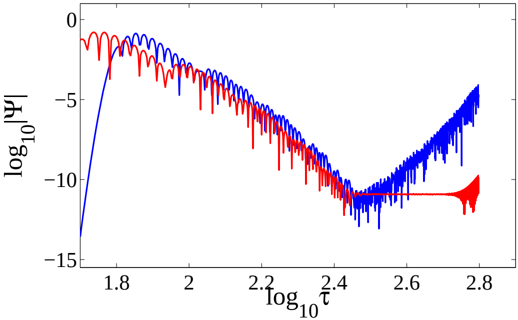


Figure: Case I: Late time decay of $|\Psi|$ at $\rho_* = 15$ (blue) and at $\rho_* = 3.64784$ (red) with the first order outflow boundary condition.

# Results: Case 2

- Lau's analytic bounday condition at $\rho = \rho_B$ for all $\tau > 0$.

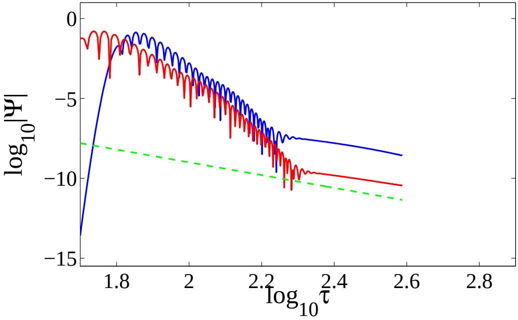


Figure: Case II: Late time decay of $\Psi$ at $\rho_* = 15$ (Blue) and at $\rho_* = 3.64784$ (Red) with Lau's NRBC. The green dashed line is the reference line with $p = -4$.

# Results: Case 3

- The spectral filtering method is applied within the spectral domain for Case II.

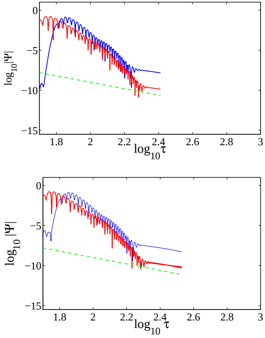


Figure: Case III: Late time decay of $\Psi$ at $\rho_* = 15$ (Blue) and at $\rho_* = 3.64784$ (Red) with Lau's NRBC and spectral filtering method. $q = 16, 12, 10, 8$ from top to bottom and left to right.

# Results: Summary

- We explain the reasons why the proper power-law decay is not obtained even with Lau's method as follows.
- The computational implementation of the first order outflow boundary condition is not *exact* even for $V = 0$. This inexactness is due to the computational inconsistency of the second order wave equation and the first order boundary condition.
- Such an inexactness produces non-physical reflections.
- To verify this, we solve the original PDE with the vanishing potential function $V = 0$.
- When $V = 0$, the PDE is a simple wave equation in $1 + 1$ space-time and the Sommerfeld condition at $\rho = \rho_B$ is theoretically exact.

# Results: Summary (cont.)

- Below is the numerical result

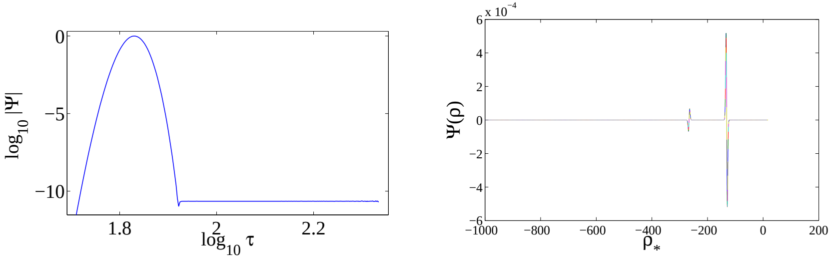


Figure: $\Psi$ with $V = 0$. The first order PDE, $\Psi_\tau + \Psi_{\rho_*} = 0$, is used as the boundary condition. $\rho_* \in [-1000, 15]$. Left: Late-time decay of $\Psi$ at $\rho = \rho_B$. Right: $\Psi$ versus $\rho_*$ at $\tau = 10^{2.3345}$.

# Conclusion and Further Work

- The second order wave equation is solved using the spectral and finite difference hybrid method
- For the right boundary, we use Lau's analytic boundary condition, which is developed by Lau [Lau2005]
- Our numerical results show that the power-law decay is obtained with the help of Lau's method even with the small value of the right boundary location. But the expected power-law of the order $p = -7$ is not obtained but the obtained order is about $p \sim -4$.
- Our numerical results show that the filtering method does not help to improve the result either.

# Conclusion and Further Work

- We explain that the reason is because the first order advection equation as the outflow boundary condition for the second order wave equation is not exact computationally even though we have the vanishing potential function.
- Lau's method helps to suppress such an inexactness of the Sommerfeld boundary condition but does not seem to be enough.
- Possible remedy: Converting the equation into a system of equations or calculating with higher precision than double precision would help to fix this problem.

# Conclusion and Further Work

Thank You!