These notes cover examples from the lecture on § 9.4 - Vectors, and § 9.5 - The Dot Product, as well as some extra examples. These cover the most important types you are likely to see. It is strongly recommended that you work more problems similar to these in order to get good at these types of problems as they are very likely to show up on quizzes and tests.

1. § 9.4 - Vectors

Definition 1.1. A vector is a “quantity” that has both magnitude and direction. We will use the notion of an arrow with some length (length defines the magnitude) pointing in a certain direction to represent a vector.

Definition 1.2. Vectors have the following properties:
Commutative: \( v + w = w + v \)
Associative: \( u + (v + w) = (u + v) + w \)
Zero Vector: \( v + 0 = 0 + v = v \)
Additive Inverse: \( v + (-v) = 0 \)

Definition 1.3. If \( \alpha \) is a scalar (in our case, a real number), and \( v \) is a vector, then a scalar multiple, which is \( \alpha v \), is defined different depending on \( \alpha \).
(a) If \( \alpha > 0 \), then the vector \( v \) changes in magnitude by \( \alpha \) and the vector is pointing in the same direction. For magnitude: If \( 0 < \alpha < 1 \), the vector becomes shorter. If \( \alpha = 1 \), the vector remains the same length. If \( \alpha > 1 \), the vector becomes longer.
(b) If \( \alpha < 0 \), then the vector \( v \) changes in magnitude by \( \alpha \) and the vector is pointing in the opposite direction. For magnitude: If \( 0 > \alpha > -1 \), the vector becomes shorter. If \( \alpha = -1 \), the vector remains the same length. If \( \alpha < -1 \), the vector becomes longer.
(c) If \( \alpha = 0 \), or \( v = 0 \), then \( \alpha v = 0 \).

Definition 1.4. Scalar multiples have the following properties:
\[
\begin{align*}
0v &= 0 \\
1v &= v \\
-1v &= -v \\
(\alpha + \beta)v &= \alpha v + \beta v \\
\alpha(v + w) &= \alpha v + \alpha w \\
\alpha(\beta v) &= (\alpha\beta)v
\end{align*}
\]

Definition 1.5. The symbol \( ||v|| \) denotes the magnitude of a vector \( v \). It has the following properties:
\[
\begin{align*}
||v|| &\geq 0 \\
||v|| &= 0 \text{ if and only if } v = 0 \\
||-v|| &= ||v|| \\
||\alpha v|| &= ||v||
\end{align*}
\]

Definition 1.6. An algebraic vector, say \( v \), is given by \( v = (a, b) \), where \( a \) and \( b \) are points in the plane. If \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) and the vector points from \( P_1 \) to \( P_2 \), then \( v = (x_2 - x_1, y_2 - y_1) \).

NOTE: If we have \( (a, b) \), then to draw this vector, start at the origin. Then go \( a \) units in the \( x \) direction and \( b \) units in the \( y \) direction. Then connect the origin to that point with an arrow.
Definition 1.7. Here are some properties for vector addition (i and j represent the $i^{th}$ and $j^{th}$ components, or $x$ and $y$ components). Let \( v = a_i + b_j = \langle a_1, b_1 \rangle \) and \( w = a_2 i + b_2 j = \langle a_2, b_2 \rangle \), and \( \alpha \) be a scalar. Then we have

\[
\begin{align*}
  v + w &= (a_1 + a_2) i + (b_1 + b_2) j = \langle a_1 + a_2, b_1 + b_2 \rangle \\
  v - w &= (a_1 - a_2) i + (b_1 - b_2) j = \langle a_1 - a_2, b_1 - b_2 \rangle \\
  \alpha v &= (\alpha a_1) i + (\alpha b_1) j = \langle \alpha a_1, \alpha b_1 \rangle \\
  ||v|| &= \sqrt{a_1^2 + b_1^2}
\end{align*}
\]

Example 1. If \( v = \langle 2, 3 \rangle \) and \( w = \langle 3, -4 \rangle \). Find \( v + w, \ v - w, \ ||v||, \ \text{and} \ ||w||. \)

Example 2. If \( v = \langle 3, 4 \rangle \) and \( w = \langle -1, 3 \rangle \). Find \( v + w, \ v - w, \ ||v||, \ \text{and} \ ||w||. \)

Definition 1.8. The unit vector is defined to be

\[ v = \frac{v}{||v||} \]

To find the magnitude of a vector \( v = \langle a, b \rangle \), all that is required is to compute \( ||v|| \). The direction, \( \theta \), which is an angle in the plane, can be found by using the formula

\[ \tan(\theta) = \frac{b}{a} \]

NOTE: When you get to the step of \( \tan(\theta) = z \), where \( z \) is a number, ask yourself which value of \( \theta \) in \([0, 2\pi]\) gives you the value of \( z \). I think this is the easiest way to understand this. Use the unit circle and draw the picture of the graph. The angle you get should make sense with the picture you draw.

Example 3. Find the direction \( \theta \) of \( v = \langle 4, -4 \rangle \).

Example 4. Find the direction \( \theta \) of \( v = \langle 1, \sqrt{3} \rangle \).

Example 5. Find the direction \( \theta \) of \( v = \langle 3, 3 \rangle \).

Example 6. Find the direction \( \theta \) of \( v = \langle -3\sqrt{3}, 3 \rangle \).

Definition 1.9. Let \( v \) be a vector. If we are given the magnitude of \( v \) (this is \( ||v|| \)) and a direction angle \( \theta \), then we can find what the vector looks like using the following formula

\[ v = ||v|| \langle \cos(\theta), \sin(\theta) \rangle \]

Example 7. If the magnitude of a vector \( v \) is 10 and has direction angle \( \theta = \frac{\pi}{3} \), write the vector \( v \) in component form.

Example 8. If the magnitude of a vector \( v \) is 4 and has direction angle \( \theta = \frac{2\pi}{3} \), write the vector \( v \) in component form.

Example 9. If the magnitude of a vector \( v \) is 8 and has direction angle \( \theta = \frac{5\pi}{6} \), write the vector \( v \) in component form.

Example 10. If the magnitude of a vector \( v \) is 5 and has direction angle \( \theta = \frac{3\pi}{4} \), write the vector \( v \) in component form.
2. § 9.5 - Dot Product

**Definition 2.1.** Let \( v = a_1 i + b_1 j \) and \( w = a_2 i + b_2 j \). Then the **dot product** is defined by \( v \cdot w \) is
\[
v \cdot w = a_1 a_2 + b_1 b_2\]

Here are some properties for the dot product

**Definition 2.2.**
\[
\begin{align*}
v \cdot w &= u \cdot v \\
u \cdot (v + w) &= u \cdot v + u \cdot w \\
u \cdot u &= ||u||^2 \\
0 \cdot u &= 0
\end{align*}
\]

**Definition 2.3.** *(Angle between vectors)* - If \( u \) and \( v \) are non-zero vectors, the angle \( \theta \), such that \( 0 \leq \theta \leq \pi \), between the vectors is defined to be
\[
\cos(\theta) = \frac{u \cdot v}{||u|| ||v||}
\]

**Definition 2.4.** If the angle \( \theta \) between two vectors is 0 using the above formula, then the vectors are **parallel**. If the angle \( \theta \) between two vectors is \( \frac{\pi}{2} \) using the above formula, then the vectors are **perpendicular** or **orthogonal**.

**Example 11.** Find the dot product of \( u \cdot v \) and the angle between the vectors for \( u = i - j \) and \( v = i + j \).

**Example 12.** Find the dot product of \( u \cdot v \) and the angle between the vectors for \( u = i - j \) and \( v = -i + j \).

**Example 13.** Find the dot product of \( u \cdot v \) and the angle between the vectors for \( u = 2i - j \) and \( v = i - 2j \).

**Example 14.** Find the dot product of \( u \cdot v \) and the angle between the vectors for \( u = \sqrt{3}i - j \) and \( v = i + j \).