These notes cover examples from the lecture on § 11.5 - Partial Fraction Decomposition, as well as some extra examples. These cover the most important types you are likely to see. It is strongly recommended that you work more problems similar to these in order to get good at these types of problems as they are very likely to show up on quizzes and tests.

1. § 11.5 - Partial Fraction Decomposition

The process of partial fraction decomposition is used to break down a complex fraction expression into a sum of fractions. For example we will learn how to break down an expression like the following

\[
\frac{5x - 1}{x^2 + x - 12} = \frac{3}{x + 4} + \frac{2}{x - 3}
\]

Each fraction has the form of \( \frac{P}{Q} \). There are 4 different cases depending on what the polynomial \( Q \) looks like. The polynomial \( Q \) can have (a) Non-repeated Linear Factors, (b) Repeated Linear Factors, (c) Non-repeated Irreducible Quadratic Factor, and (d) Repeated Irreducible Quadratic Factor. We will consider these cases individually. Below there is one worked example of each kind of problem and each section has some example problems that you can work out.

1.1. Non-repeated Linear Factors. If \( Q \) has only non-repeated linear factors, then \( Q \) looks like

\[
Q(x) = (x - a_1)(x - a_2)\ldots(x - a_n)
\]

where \( a_1, a_2, \ldots, a_n \) are roots of the polynomial \( Q \). Then we can break up the fraction as

\[
P(x) \quad Q(x) = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \ldots + \frac{A_n}{x - a_n}
\]

where \( A_1, A_2, \ldots, A_n \) are expressions we need to find.

Example 1. Find the partial fraction decomposition of

\[
\frac{x}{x^2 - 5x + 6}
\]

Solution: First we have to decompose the denominator into pieces:

\[
x^2 - 5x + 6 = (x - 2)(x - 3)
\]

Then we rewrite the expression in the form above and find a common denominator

\[
\frac{x}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}
\]

Now we can set the numerators equal to get

\[
x = A(x - 3) + B(x - 2)
\]

The easiest way to solve for \( A \) and \( B \) is by plugging “special” numbers into the above equation. Notice that if we plug in \( x = 3 \) and \( x = 2 \), we easily get the values of \( A \) and \( B \). Here is the work:

\[
x = A(x - 3) + B(x - 2)
\]

\[
3 = A(3 - 3) + B(3 - 2)
\]

\[
3 = B
\]
and
\[
x = A(x - 3) + B(x - 2) \\
2 = A(2 - 3) + B(2 - 2) \\
2 = -A \\
-2 = A
\]
So then we have the final expression
\[
\frac{x}{x^2 - 5x + 6} = \frac{-2}{x - 2} + \frac{3}{x - 3}
\]

Example 2. Find the partial fraction decomposition of
\[
\frac{3x}{(x + 2)(x - 1)}
\]
Example 3. Find the partial fraction decomposition of
\[
\frac{x}{(x - 2)(x - 1)}
\]
Example 4. Find the partial fraction decomposition of
\[
\frac{3x}{(x + 2)(x - 4)}
\]
Example 5. Find the partial fraction decomposition of
\[
\frac{4}{2x^2 - 5x - 3}
\]

1.2. Repeated Linear Factors. If \(Q\) has a repeated linear factor, then \(Q\) has some term of the form
\[\frac{P(x)}{Q(x)} = \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_n}{(x - a)^n}\]
where \(a\) is a root and \(n \geq 2\). Then we can break up the fraction as

Example 6. Find the partial fraction decomposition of
\[
\frac{x + 2}{x^3 - 2x^2 + x}
\]
Solution: First we have to decompose the denominator into pieces:
\[x^3 - 2x^2 + x = x(x - 1)^2\]
Then we rewrite the expression in the form above along with the first term that comes from the case we already did. Then we find a common denominator and get
\[
\frac{x + 2}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}
\]
\[
\frac{x + 2}{x^3 - 2x^2 + x} = \frac{A(x - 1)^2 + Bx(x - 1) + Cx}{x(x - 1)^2}
\]
Now we can set the numerators equal to get
\[x + 2 = A(x - 1)^2 + Bx(x - 1) + Cx\]
The easiest way to solve for A, B, and C is by plugging “special” numbers into the above equation as before. Notice that if we plug in \(x = 0\) and \(x = 1\), we easily get the values of A and C. Here is the work:

\[
x + 2 = A(x - 1)^2 + Bx(x - 1) + Cx
\]

\[
0 + 2 = A(0 - 1)^2 + B(0)(0 - 1) + C(0)
\]

\[
2 = A
\]

and

\[
x + 2 = A(x - 1)^2 + Bx(x - 1) + Cx
\]

\[
1 + 2 = A(1 - 1)^2 + B(1)(1 - 1) + C(1)
\]

\[
3 = C
\]

Now so far we have

\[
x + 2 = 2(x - 1)^2 + Bx(x - 1) + 3x
\]

At this point, we can plug in ANY value for \(x\) (except 0 and 1 of course) and solve for B. We choose 2, so then

\[
x + 2 = 2(x - 1)^2 + Bx(x - 1) + 3x
\]

\[
2 + 2 = 2(2 - 1)^2 + B(2)(2 - 1) + 3(2)
\]

\[
4 = 8 + 2B
\]

\[-2 = B
\]

So then we have the final expression

\[
\frac{x + 2}{x^3 - 2x^2 + x} = \frac{2}{x} + \frac{-2}{x - 1} + \frac{3}{(x - 1)^2}
\]

Example 7. Find the partial fraction decomposition of

\[
\frac{x + 1}{x^2(x - 2)}
\]

Example 8. Find the partial fraction decomposition of

\[
\frac{x - 3}{(x + 2)(x + 1)^2}
\]

Example 9. Find the partial fraction decomposition of

\[
\frac{x^2 + x}{(x + 2)(x - 1)^2}
\]

Example 10. Find the partial fraction decomposition of

\[
\frac{x^2}{(x - 1)^2(x + 1)^2}
\]

1.3. Irreducible Quadratic Factor. If Q has an irreducible quadratic factor, then Q has some term of the form \(ax^2 + bx + c\) that we cannot break down any further. In this case we attempt to use the form

\[
\frac{Ax + B}{ax^2 + bx + c}
\]

where A and B are expressions we need to find.

Example 11. Find the partial fraction decomposition of

\[
\frac{3x - 5}{x^3 - 1}
\]
Solution: As before, the first step is to reduce the denominator as much as we can. This means

\[x^3 - 1 = (x - 1)(x^2 + x + 1)\]

We can find this easily by noticing that \(x = 1\) is a root of \(x^3 - 1\), so we can do synthetic division to get the other polynomial. The quadratic term above is irreducible, so we cannot break this up any further. Now we can use the definition above to write out the expression

\[
\frac{3x - 5}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}
\]

where the first term comes from Case 1.1 above and the second part from the definition we defined here in 1.3. Then we follow the same procedure as above to get

\[3x - 5 = A(x^2 + x + 1) + (Bx + C)(x - 1)\]

by finding the common denominator and setting the numerators equal. Here, we can also be clever in choosing values of \(x\) to plug in to easily solve for the \(A\), \(B\), and \(C\). Notice that if we pick \(x = 1\), we get

\[
3x - 5 = A(1^2 + 1 + 1) + (B(1) + C)(1 - 1)
\]

\[
-2 = 3A
\]

\[
A = -\frac{2}{3}
\]

Now we have a value for \(A\), so we have

\[3x - 5 = -\frac{2}{3}(x^2 + x + 1) + (Bx + C)(x - 1)\]

Now notice that if we choose \(x = 0\), we get

\[3x - 5 = -\frac{2}{3}(0^2 + 0 + 1) + (B(0) + C)(0 - 1)
\]

\[
-5 = -\frac{2}{3} - C
\]

\[
C = 5 - \frac{2}{3}
\]

\[
C = \frac{13}{3}
\]

So far our expression is

\[3x - 5 = -\frac{2}{3}(x^2 + x + 1) + (Bx + \frac{13}{3})(x - 1)\]

Now we have reached the point where we can plug in ANY value for \(x\) (not the ones we already chose) to solve for \(B\). So let’s pick \(x = 2\). Then we get

\[3x - 5 = -\frac{2}{3}(2^2 + 2 + 1) + (B(2) + \frac{13}{3})(2 - 1)
\]

\[
3(2) - 5 = -\frac{2}{3}(7) + (2B + \frac{13}{3})(1)
\]

\[
1 = -\frac{14}{3} + 2B + \frac{13}{3}
\]

\[
1 = -\frac{1}{3} + 2B
\]

\[
2B = \frac{4}{3}
\]

\[
B = \frac{2}{3}
\]
So now we have all the values, and our final solution has the form
\[
\frac{3x - 5}{x^3 - 1} = \frac{-2}{3} \frac{2}{x - 1} + \frac{2}{3} \frac{13}{x^2 + x + 1}
\]

Example 12. Find the partial fraction decomposition of
\[
\frac{1}{x^2 + 1}
\]

Example 13. Find the partial fraction decomposition of
\[
\frac{1}{(x + 1)(x^2 + 4)}
\]

Example 14. Find the partial fraction decomposition of
\[
\frac{x + 4}{x^2(x^2 + 4)}
\]

Example 15. Find the partial fraction decomposition of
\[
\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)}
\]

1.4. Repeated Irreducible Quadratic Factor. If \( Q \) has a repeated irreducible quadratic factor, then \( Q \) has some term of the form
\[
(ax^2 + bx + c)^n
\]
and \( n \geq 2 \). Then we attempt to use the form
\[
\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}
\]
where \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) are expressions we need to find.

Example 16. Find the partial fraction decomposition of
\[
\frac{x^3 + x^2}{(x^2 + 4)^2}
\]

Solution: Using the above definition to rewrite our expression, we get
\[
\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}
\]

Finding a common denominator, and setting the numerator equal, we get
\[
x^3 + x^2 = (Ax + B)(x^2 + 4) + Cx + D
\]
\[
x^3 + x^2 = Ax^3 + Bx^2 + 4Ax + 4B + Cx + D
\]
\[
x^3 + x^2 = Ax^3 + Bx^2 + (4A + C)x + D
\]

Now we set the coefficients on the left and right hand sides equal. By that we mean, the coefficients of the \( x^3 \) terms must be the same, the coefficients of the \( x^2 \) terms must be the same, and so on. Then we get the system
\[
\begin{align*}
A &= 1 \\
B &= 1 \\
4A + C &= 0 \\
4B + D &= 0
\end{align*}
\]
Since we already know what $A$ and $B$ are, we can use them to solve for $C$ and $D$ using equations 3 and 4 above. So now we just have to solve

$$\begin{aligned}
4 + C &= 0 \\
4 + D &= 0
\end{aligned}$$

From this, it is clear that $C = -4$ and $D = -4$. So then our final solution is

$$\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{x + 1}{x^2 + 4} + \frac{-4x - 4}{(x^2 + 4)^2}$$

\[\Box\]

**Example 17.** Find the partial fraction decomposition of

$$\frac{x^2 + 2x + 3}{(x^2 + 4)^2}$$

**Example 18.** Find the partial fraction decomposition of

$$\frac{(x^3 + 1)}{(x^2 + 16)^2}$$

**Example 19.** Find the partial fraction decomposition of

$$\frac{x^3}{(x^2 + 16)^3}$$

**Example 20.** Find the partial fraction decomposition of

$$\frac{x^2}{(x^2 + 4)^3}$$