Laplace's Equation

From Boyce + DiPrima Section 10.8

2) \( \Delta u = u_{xx} + u_{yy} = 0 \)

\( u(0,y) = 0 \quad u(x,0) = h(x) \)

\( u(a,y) = 0 \quad u(x,b) = 0 \)

Solution: Assume a product solution: \( u(x,y) = \phi(x) \psi(y) \)

Now use separation of variables:

\[ \phi''(x) \psi(y) + \phi(x) \psi''(y) = 0 \]

\[ \Rightarrow \frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = -\lambda \]

\[ \Rightarrow \phi''(x) + \lambda \phi(x) = 0 \quad \text{and} \quad \psi''(y) - \lambda \psi(y) = 0 \]

The ODE in terms of \( x \) is the usual eigenvalue problem. We work with this one first since its BC's are homogeneous.

\( u(0,y) = 0 \Rightarrow \phi(0) = 0 \)

\( u(a,y) = 0 \Rightarrow \phi(a) = 0 \)

\[ \Rightarrow \phi(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x \]

\[ \sqrt{\lambda} = \frac{n\pi}{a} \]

For the ODE in terms of \( y \), we have to use the fact that Laplace's Equation is translation invariant to solve this problem. Since we have \( -\lambda \) in \( \psi'' - \lambda \psi = 0 \), we will get

\[ \psi(y) = d_1 \cosh \left( \frac{n\pi}{a} y \right) + d_2 \sinh \left( \frac{n\pi}{a} y \right) \]

Since \( \Delta u \) is translation invariant (shifting \( y \) by a constant) the translated \( \psi \) will still satisfy the PDE

\[ \Rightarrow \psi(y) = d_1 \cosh \left( \frac{n\pi}{a} (b-y) \right) + d_2 \sinh \left( \frac{n\pi}{a} (b-y) \right) \]

also solves the ODE.
The condition $u(x, b) = 0 \Rightarrow \psi(b) = 0$. This is why we need to shift our solution, so the boundary conditions work. So then you get that

$$\psi(y) = \alpha \sinh \left( \frac{n \pi b}{a} (b-y) \right)$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi x}{a} \right) \sinh \left( \frac{n \pi b}{a} (b-y) \right) \quad \Box$$

Now the last BC will give us $A_n$ formula, since

$$u(x, 0) = h(x) = \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n \pi b}{a} \right) \sin \left( \frac{n \pi x}{a} \right) \quad \star$$

Note that $\star$ is a constant since $A_n, \sin \left( \frac{n \pi b}{a} \right)$ are constant. So this is just the Fourier sine series for $h(x)$. So as before with the wave equation, we set all of $\star$ equal to the coefficient formula,

$$A_n \sinh \left( \frac{n \pi b}{a} \right) = \frac{2}{a} \int_{0}^{a} h(x) \sin \left( \frac{n \pi x}{a} \right) dx$$

$$\Rightarrow A_n = \frac{2}{a \sinh \left( \frac{n \pi b}{a} \right)} \int_{0}^{a} h(x) \sin \left( \frac{n \pi x}{a} \right) dx \quad \dagger$$

and $\Box$ is the solution with $A_n$ defined as $\dagger$.

It may be pertinent to attempt additional questions from 10.8 to practice your skills solving these PDE for the final.