1. §5.3 - # 1, 2, 5, 8

For questions 1 and 2, it will be helpful for you to read the bottom half of page 265. You will have to work the same method up to Equation 3 on that page, then use the given information from the problem to calculate $\phi''(x_0), \phi'''(x_0), \phi^{(4)}(x_0)$. The main equations you will need are:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$
$$\phi''(x) = -\frac{Q(x)}{P(x)}\phi'(x) - \frac{R(x)}{P(x)}\phi(x)$$

1) The differential equation we have is $y'' + xy' + y = 0$ with $y(0) = 1$ and $y'(0) = 0$. So in our case, it is clear to see that $P(x) = 1, Q(x) = x, R(x) = 1$. So we get the following general formulas for our $\phi$ functions

$$\phi''(x) = -x\phi'(x) - \phi(x)$$
$$\phi'''(x) = -2\phi'(x) - x\phi''(x)$$
$$\phi^{(4)}(x) = -3\phi''(x) - x\phi'''(x)$$

So using the fact that $y(0) = \phi(0) = 1$ and $y'(0) = \phi'(0) = 0$, we get the desired result

$$\phi''(0) = -1$$
$$\phi'''(0) = 0$$
$$\phi^{(4)}(0) = 3$$

2) The differential equation we have is $y'' + \sin(x)y' + \cos(x)y = 0$ with $y(0) = 0$ and $y'(0) = 1$. So in our case, it is clear to see that $P(x) = 1, Q(x) = \sin(x), R(x) = \cos(x)$. So we get the following general formulas for our $\phi$ functions

$$\phi''(x) = -\sin(x)\phi'(x) - \cos(x)\phi(x)$$
$$\phi'''(x) = -2\cos(x)\phi'(x) - \sin(x)\phi''(x) + \sin(x)\phi(x)$$
$$\phi^{(4)}(x) = \cos(x)\phi(x) + \sin(x)\phi'(x) - 3\cos(x)\phi''(x) - \sin(x)\phi'''(x)$$

So using the fact that $y(0) = \phi(0) = 0$ and $y'(0) = \phi'(0) = 1$, we get the desired result

$$\phi''(0) = 0$$
$$\phi'''(0) = -2$$
$$\phi^{(4)}(0) = 0$$

5) We have $y'' + 4y' + 6xy = 0$ and $x_0 = 0, x_0 = 4$. All we have to observe is that in usual form of $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$, that

$$\frac{Q(x)}{P(x)} = 4, \quad \frac{R(x)}{P(x)} = 6x$$

Both of these equations have no singularities (undefined points), so the largest radius of convergence of the power series solution is $\rho = \infty$ at both $x_0 = 0$ and $x_0 = 4$.

8) We have $xy'' + y = 0$ and $x_0 = 1$. All we have to observe is that in usual form of $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$, that
$$\frac{Q(x)}{P(x)} = 0 \quad \frac{R(x)}{P(x)} = \frac{1}{x}$$

We see that the second equation has a singularity at the point $x = 0$. So since we are centered about the point $x_0 = 1$, the largest circle we can draw centered at 0 is of radius 1 before we get to the singularity. Therefore, we have that $\rho = 1$.

2. §5.4 - # 1, 2, 5, 6, 22, 26

Questions 1, 2, 5, 6 all involve using the substitution $y(x) = x^r$. The key is then to solve for what $r$ is, and it will be one of three cases: 1) distinct real roots, 2) equal real roots, and 3) complex roots. The general formulas for each are outlined on p. 273-4.

1) Using the substitution, we get

$$x^2y'' + 4xy' + 2y = 0$$
$$x^2r(r - 1)x^{r-2} + 4rx^{r-1} + 2x^r = 0$$
$$r(r - 1)x^r + 4rx^r + 2x^r = 0$$
$$[r^2 + 3r + 2]x^r = 0$$
$$r^2 + 3r + 2 = 0$$
$$r = -1, -2$$
$$\Rightarrow y(x) = c_1x^{-1} + c_2x^{-2}$$

2) Using the substitution $y = (x + 1)^r$, we get

$$(x + 1)^2y'' + 3(x + 1)y' + \frac{3}{4}y = 0$$
$$(x + 1)^2(r(r - 1))(x + 1)^{r-2} + 3(x + 1)r(x + 1)^{r-1} + \frac{3}{4}(x + 1)^r = 0$$
$$[r^2 + 2r + \frac{3}{4}] (x + 1)^r = 0$$
$$r^2 + 2r + \frac{3}{4} = 0$$
$$r = -\frac{1}{2}, -\frac{3}{2}$$
$$\Rightarrow y(x) = c_1|x + 1|^{-\frac{1}{2}} + c_2|x + 1|^{-\frac{3}{2}}$$

5) Using the substitution $y = x^r$, we get

$$x^2y'' - xy' + y = 0$$
$$[r^2 - 2r + 1]x^r = 0$$
$$(r - 1)^2 = 0$$
$$r = 1, 1$$
$$\Rightarrow y(x) = (c_1 + c_2 \ln |x|)x$$
6) Using the substitution $y = (x - 1)^r$, we get

\[(x - 1)^2 y'' + 8(x - 1)y' + 12y = 0\]

\[r^2 + 7r + 12 = 0\]

\[r = -4, -3\]

\[\Rightarrow y(x) = c_1(x - 1)^{-4} + c_2(x - 1)^{-3}\]

For problems 22 and 26, we have to classify if the singularity is regular or irregular. According to the text, to check this, we have to have the following limits be finite at the points which are singularities, say $x_0$.

\[\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}\]

22) The ODE is $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$. If we divide by $P(x) = x^2$, we see that the singular point is $x = 0$. Now computing the limits

\[\lim_{x \to 0} \frac{x}{x^2} = 1 < \infty\]

\[\lim_{x \to 0} \frac{x^2 (x^2 - \nu^2)}{x^2} = -\nu^2 < \infty\]

So since both of the limits are finite, we have that $x = 0$ is a regular singular point.

22) The ODE is $x(3 - x)y'' + (x + 1)y' - 2y = 0$. If we divide by $P(x) = x^2$, we see that the singular points are $x = 0$ and $x = 3$. Now computing the limits

\[\lim_{x \to 0} \frac{x + 1}{x(3 - x)} = \frac{4}{3} < \infty\]

\[\lim_{x \to 0} \frac{-2}{x(3 - x)} = 0 < \infty\]

and

\[\lim_{x \to 3} (x - 3) \frac{x + 1}{x(3 - x)} = -\frac{4}{3} < \infty\]

\[\lim_{x \to 3} (x - 3)^2 \frac{-2}{x(3 - x)} = 0 < \infty\]

So since all of the limits are finite, we have that $x = 0$ and $x = 3$ are regular singular points.