A Local Discontinuous Galerkin Method for the Coupled BBM-System

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Outline

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abcd Boussinesq Equations and the Coupled BBM-System

Overview of the Work

- Derivation and implementation of a Discontinuous Galerkin (DG) method to solve a system of coupled nonlinear hyperbolic partial differential equations.
- Specifically we look at the couple BBM-system, which is a special case of the abcd-Boussinesq system.
- Develop numerical fluxes that conserve energy and work well for long time simulations.
- Prove stability results and error estimates for the proposed method.
- Numerically test the method using convergence tests, solitary wave generation and solitary wave collisions.

Applications of the abcd Boussinesq System

- The Boussinesq system of equations are used to model propagation of long-crested waves on large lakes or on the ocean.
- The equations can also model water waves moving through a channel which have small amplitude and long wavelengths.
- Models in coastal engineering utilize Boussinesq-type equations to simulate water waves in shallow waters and harbors.

abcd Boussinesq System Derivation

Steps *(J. Bona, M. Chen, J. Saut (2002))*:

- Asymptotic expansion of the Euler equations
- Assumptions: small amplitude waves with long wavelength
- Let h be approximate constant depth of a channel, A be the wave amplitude, and λ be wavelength
- $\alpha = \frac{A}{h} \ll 1$, and $\beta = \frac{h^2}{\lambda^2} \ll 1$
- abcd Boussinesq system

$$\eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0,$$
$$u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0,$$

where $u(x,t)$ is the horizontal velocity of the fluid at the scaled height $\sqrt{\frac{2}{3}}h$ below the undisturbed surface.

abcd Boussinesq System Parameters

- The parameters for the abcd Boussinesq system have the following relationships:

$$a+b=\frac{1}{2}\left(\theta^2-\frac{1}{3}\right), \quad c+d=\frac{1}{2}(1-\theta^2)\geq 0,$$
$$a+b+c+d=\frac{1}{3},$$

  where $\theta \in [0,1]$ specifies the scaled height for horizontal velocity variable, $u(x,t)$.

- With $\theta^2=\frac{2}{3}(b=d=\frac{1}{6})$, and $a=c=0$, we obtain the coupled BBM system

$$\eta_t+u_x+(\eta u)_x-\frac{1}{6}\eta_{xxt}=0,$$
$$u_t+\eta_x+uu_x-\frac{1}{6}u_{xxt}=0.$$

Coupled-BBM System and Single BBM equation

- Benjamin, Bona, and Mahony (hence BBM) published the results for the BBM equation in 1972.
- Single BBM equation

$$v_t + v_x + v v_x - \frac{1}{6} v_{xxt} = 0$$

- Equations as an improvement of the KdV equation for modeling long surface gravity waves of small amplitude.

$$v_t + 6 v v_x + v_{xxx} = 0 \quad \text{(KdV)}$$

- BBM equations are stable at high wave numbers, whereas the KdV equation is unstable for high wave numbers.
- Single BBM equation is a simplification of the coupled BBM-system, in that the single BBM equation assumes unidirectional wave motion.

Coupled-BBM System Conserved Quantities

The following quantities are conserved by the coupled-BBM system:

- $\displaystyle\int_{\mathbb{R}} \eta \; dx$, and $\displaystyle\int_{\mathbb{R}} u \; dx$
- $\displaystyle\int_{\mathbb{R}} (\eta u + \eta_x u_x) \; dx$
- $\displaystyle\frac{1}{2}\int_{\mathbb{R}} \left[\eta^2 + (1+\eta)u^2\right] \; dx$

In numerical tests, the last quantity, denoted as

$$\mathcal{H}(\eta, u, t) = \frac{1}{2}\int_{\mathbb{R}} \left[\eta^2 + (1+\eta)u^2\right] \; dx$$

will be the Hamiltonian functional we will conserve numerically.

Previous Work on the Coupled BBM System

J.Bona, M.Chen (1998)

- Existence, uniqueness, and regularity of the solution to the coupled BBM system is given.
- Numerical method using a finite difference scheme, and predictor-corrector methods.
- The scheme is fourth-order accurate in both space and time.
- The scheme is unconditionally stable.
- Numerical tests for generation of solitary waves, head-on collision of solitary waves.

A. Alazman, J. Albert, J. Bona, M. Chen, J. Wu (2006)

- Comparison between the coupled BBM-system and single BBM-equation.
- Approximate traveling wave solutions are generated from exact solution to single BBM equation.
- The numerical method for the BBM-system used is that in *J. Bona, M. Chen (1998)*.
- Numerical tests for generation of solitary waves, dispersive waves, solitary wave interactions.

V. Dougalis, D. Mitsotakis, J. Saut (2009), (2010)

- Studies the multi-dimensional version of the BBM-system
- A standard finite element Galerkin method for 2D case, with continuous piecewise linear elements on a triangular mesh. A second order RK scheme is used in time.
- Method achieves optimal convergence.
- Numerical tests for solitary waves passing an obstacle (pier), without obstacle (shoreline), and with an elliptic “island” for a earthquake-generated tsunami wave.
- (2010) is an extension of (2009) with a modified Galerkin method, more applications.

LDG Method for the BBM-System

Coupled BBM-system in Conservation Form

The coupled BBM-system given by

$$\begin{cases} \eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x - \frac{1}{6}u_{xxt} = 0. \end{cases}$$

Coupled BBM-system in Conservation Form

The coupled BBM-system given by

$$\begin{cases} \eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x - \frac{1}{6}u_{xxt} = 0. \end{cases}$$

We can write the above system in a conservation form

$$\begin{cases} \left(\eta - \frac{1}{6}\eta_{xx}\right)_t + (u + (\eta u))_x = 0, \\ \left(u - \frac{1}{6}u_{xx}\right)_t + \left(\eta + \frac{u^2}{2}\right)_x = 0. \end{cases}$$

Coupled BBM-system as a system of first order equations

We can rewrite the coupled-BBM system into a system of first order equations as the following

$$
\begin{aligned}
w_t + (\eta + q)_x &= 0 \\
w &= u - \frac{1}{6} r_x \\
r &= u_x \\
q &= \frac{1}{2} u^2 \\
v_t + (u + p)_x &= 0 \\
v &= \eta - \frac{1}{6} s_x \\
s &= \eta_x \\
p &= \eta u
\end{aligned}
$$

DG Formulation

The DG method is formulated as follows: for any test functions $\phi_h, \psi_h, \varphi_h, \zeta_h, \rho_h, \theta_h, \xi_h, \vartheta_h \in V_h^k$, find $w_h, v_h, u_h, \eta_h, r_h, s_h, p_h, q_h \in V_h^k$ such that

$$\int (w_h)_t \phi_h \, dx - \int (\eta_h + q_h)(\phi_h)_x \, dx - \sum_{j=1}^{N} ((\tilde{\eta}_h + \widehat{q_h})[\phi_h])_{j+\frac{1}{2}} = 0$$

$$\int w_h \psi_h \, dx - \int u_h (\psi_h)_x \, dx - \frac{1}{6} \int r_h (\psi_h)_x \, dx - \frac{1}{6} \sum_{j=1}^{N} (\widehat{r_h}[\psi_h])_{j+\frac{1}{2}} = 0$$

$$\int r_h \varphi_h \, dx + \int u_h (\varphi_h)_x \, dx + \sum_{j=1}^{N} (\widehat{u_h}[\varphi_h])_{j+\frac{1}{2}} = 0$$

$$\int q_h \zeta_h \, dx - \int \left(\frac{1}{2}(u_h)^2\right) \zeta_h \, dx = 0$$

DG Formulation (cont.)

$$\int (v_h)_t \rho_h \, dx - \int (u_h + p_h)\,(\rho_h)_x \, dx - \sum_{j=1}^{N} ((\tilde{u}_h + \widehat{p_h})[\rho_h])_{j+\frac{1}{2}} = 0$$

$$\int v_h \theta_h \, dx - \int \eta_h \theta_h \, dx - \frac{1}{6} \int s_h (\theta_h)_x \, dx - \frac{1}{6} \sum_{j=1}^{N} (\widehat{s_h}[\theta_h])_{j+\frac{1}{2}} = 0$$

$$\int s_h \xi_h \, dx + \int \eta_h (\xi_h)_x \, dx + \sum_{j=1}^{N} (\widehat{\eta_h}[\xi_h])_{j+\frac{1}{2}} = 0$$

$$\int p_h \vartheta_h \, dx - \int (\eta_h u_h) \vartheta_h \, dx = 0$$

Choice of Numerical Flux

We investigate two different choices of numerical flux, depending on what properties we wish to preserve. First is the *alternating flux*

$$\begin{cases} \widehat{u_h} &= u_h^+, \\ \widehat{\eta_h} &= \eta_h^-. \end{cases} \qquad \begin{cases} \widetilde{u_h} + \widehat{p_h} &= u_h^+ + p_h^+, \\ \widetilde{\eta_h} + \widehat{q_h} &= \eta_h^- + q_h^-, \\ \widehat{r_h} &= r_h^-, \\ \widehat{s_h} &= s_h^+. \end{cases}$$

- Choice of flux follows from trying to recover the Hamiltonian functional.
- Choosing u_h, η_h, and p_h, q_h, and r_h, s_h from opposite sides, the summation terms, and some of the integrals cancel out from integration by parts.
- Remaining terms give the Hamiltonian functional which is conserved by the method.

Stability Theorem

Theorem (Stability, (Xing, B.))

For the choice of alternating flux, the Hamiltonian functional, $\mathcal{H}_h(\eta_h, u_h, t)$, is conserved by the LDG method, i.e.

$$\frac{d}{dt}\mathcal{H}_h(\eta_h, u_h, t) = \frac{d}{dt}\int_I (\eta_h^2 + (1+\eta_h)u_h^2)\ dx = 0$$

for all time.

Idea of the proof: The proof has a similar flavor to the energy conservation theorem found in *M. Chen, Y. Liu (2012)* at the PDE level. Choosing the alternating flux from the previous slides, boundary terms can be eliminated by integration by parts identities, to yield the Hamiltonian functional.

Choice of Numerical Flux

Second, is the upwind flux which introduces numerical dissipation, and has the choices of

$$\begin{cases} \widetilde{u_h} &= \{u_h\} - \frac{1}{2}[\eta_h], \\ \widetilde{\eta_h} &= \{\eta_h\} - \frac{1}{2}[u_h]. \end{cases} \qquad \begin{cases} \widehat{q_h} &= \{q_h\} - \frac{1}{2}[p_h], \\ \widehat{p_h} &= \{p_h\} - \frac{1}{2}[q_h]. \end{cases}$$

$$\begin{cases} \widetilde{(u_h)_t} &= \{(u_h)_t\} + \frac{1}{2}[(\eta_h)_t], \\ \widetilde{(\eta_h)_t} &= \{(\eta_h)_t\} + \frac{1}{2}[(u_h)_t]. \end{cases} \qquad \begin{cases} \widetilde{(r_h)_t} &= \{(r_h)_t\} - \frac{1}{2}[(s_h)_t], \\ \widetilde{(s_h)_t} &= \{(s_h)_t\} - \frac{1}{2}[(r_h)_t]. \end{cases}$$

- Notation: $\{u_h\} = \frac{u_h^+ + u_h^-}{2}$ and $[u_h] = u_h^+ - u_h^-$
- Choice of flux follows from eliminating the third derivative term to get a system of hyperbolic conservation laws
- Upwind flux is the standard choice for this type of system
- Chosen to add numerical dissipation to the system

Energy Dissipation Theorem

Theorem (Energy Dissipation, (Xing, B.))

For the choice of upwind flux, the Hamiltonian functional, $\mathcal{H}_h(\eta_h, u_h, t)$, satisfies

$$\frac{d}{dt}\mathcal{H}_h(\eta_h, u_h, t) = \frac{d}{dt}\int_I (\eta_h^2 + (1+\eta_h)u_h^2)\ dx \leq 0$$

with the LDG method.

Idea of the proof: Choosing the upwind flux choices from previous slides, not all boundary terms from the DG method are eliminated. These terms can be bounded by application of Young's inequality to get the energy decreasing property.

Advantages/Disadvantages for Numerical Fluxes

Comparison of Alternating vs. Upwind

- Alternating Flux
  - Method is stable
  - Conserves energy exactly
  - Good for long time simulations
- Upwind Flux
  - Method is stable
  - Dissipates energy over time
  - Not accurate for long time simulations
  - Better choice when shocks/discontinuities are present

Time Discretizations

We have used two different types of time discretizations over the course of the project:

- Strong Stability Preserving (SSP) Runge-Kutta (RK) Methods
  1. SSPRK4 explict time stepping method
  2. High order SSP methods maintain the total variation diminishing (TVD) property
  3. SSP methods are used to control numerical oscillations that occur around discontinuities
- Midpoint Rule Method
  1. Implicit time stepping method
  2. Conserves the discrete energy equivalent to the continuous case, over longer time than SSPRK4
  3. Computationally expensive as this is an implicit method

Remaining Work

- The error estimate proof of the LDG method for the single BBM equation case is partially completed.
- The proof would establish the sub-optimal error estimate

$$||u - u_h|| \leq Ch^{k+\frac{1}{2}}$$

  where u is the true solution, u_h is the LDG approximation, and k is the degree of the piecewise polynomial space.
- We would to prove a similar estimate for the coupled BBM-system. Difficulty arises in this proof due to the nonlinear terms present and the coupled nature of the system.

Solutions to the BBM-system (Exact Traveling Wave Solution)

- From *J. Bona, M. Chen (1998)*, the exact traveling wave solution to the BBM-system is

$$u(x,t) = 3k \operatorname{sech}^2\left(\frac{3}{\sqrt{10}}(x-kt-x_0)\right),$$

$$\eta(x,t) =$$

$$\frac{15}{4}\left(-2+\cosh\left(3\sqrt{\frac{2}{5}}(x-kt-x_0)\right)\right)\operatorname{sech}^4\left(\frac{3}{\sqrt{10}}(x-kt-x_0)\right)$$

where $k = \pm\frac{5}{2}$, and x_0 is the x value where the center of the wave is located

Solutions to the BBM-system (Approximate Solitary Wave Solution)

- From *A. Alazman, et. all (2006)*, the coupled BBM-system has solitary wave solutions similar to the single BBM equation given by

$$v_t + v_x + \frac{3}{2}\epsilon v v_x - \frac{1}{6}\epsilon v_{xxt} = 0,$$

where ϵ represents the ratio of the maximum wave amplitude to the undisturbed depth of the liquid.

- The *exact* traveling wave solution to the single BBM equation is

$$v(x,t) = \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{3}{\kappa}}(x - \kappa t - x_0)\right),$$

where $\kappa = 1 + \epsilon/2$.

Solutions to the BBM-system (Approximate Solitary Wave Solution)

- An *approximate* solitary wave can be constructed using the following initial condition with the coupled BBM-system

$$\eta(x,0) = v(x,0),$$
$$u(x,0) = v(x,0) - \frac{1}{4}\epsilon v(x,0)^2,$$

  where $v(x,t)$ is the exact traveling solution to the single BBM-equation
- Compare the single BBM solution to the coupled-BBM system with given initial data
- Approximate solitary wave for the coupled-BBM system, $\eta(x,t)$, is accurate to $\mathcal{O}\left(\frac{1}{\epsilon}\right)$ in time

Convergence Test: Alternating Flux, SSPRK4 in Time (Exact Traveling Wave Solution)

Parameters: $k = 2, L = 40, \Delta x = \frac{1}{2^j}$ for $j = 0, \ldots, 4$, $\Delta t = .1\Delta x$, $T = 1$

Nx	j	$\|e^\eta\|_{L^1}$	Order	$\|e^u\|_{L^1}$	Order
40	0	1.6003e-00		9.3584e-01	
80	1	1.5717e-01	3.34	6.9160e-02	3.75
160	2	1.5362e-02	3.35	5.0564e-03	3.77
320	3	1.7227e-03	3.15	5.2204e-04	3.27
640	4	2.0514e-04	3.06	6.4118e-05	3.02

Convergence Test: Alternating Flux, and Midpoint Rule in Time (Exact Traveling Wave Solution)

Parameters: $k = 2, L = 40, \Delta x = \frac{1}{2^j}$ for $j = 0, \ldots, 4$, $\Delta t = .1\Delta x^2$, $T = 1$, tolerance $= 10^{-10}$

Nx	j	$\|e^\eta\|_{L^1}$	Order	$\|e^u\|_{L^1}$	Order
40	0	2.1994e-00		1.5848e-00	
80	1	1.7709e-01	3.63	1.1434e-01	3.79
160	2	1.5581e-02	3.50	7.0977e-03	4.00
320	3	1.6858e-03	3.20	6.0759e-04	3.54
640	4	1.9711e-04	3.09	6.7434e-05	3.17

Convergence Test: Upwind Flux, and SSPRK4 in Time (Approximate Solitary Wave Solution)

Parameters: $k = 2, L = 40, \Delta x = \frac{1}{2^j}$ for $j = 0, \ldots, 4$, $\Delta t = .1\Delta x$, $T = 1$

Nx	j	$\|e^\eta\|_{L^1}$	Order	$\|e^u\|_{L^1}$	Order
40	0	1.6629e-00		1.0943e-00	
80	1	1.1121e-01	3.90	1.1281e-01	3.28
160	2	8.0167e-03	3.79	1.2562e-02	3.17
320	3	7.3197e-04	3.45	1.5346e-03	3.03
640	4	7.8245e-05	3.23	1.9317e-04	2.99

Long Time Test Approximation - Alternating Flux, SSPRK4 (Exact Traveling Wave Solution)

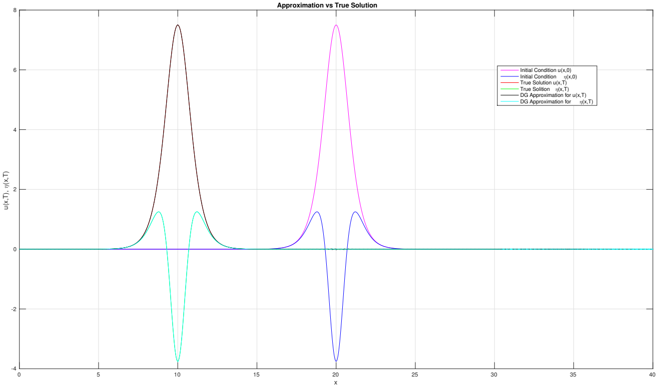

Figure : For the long time test, we run the code up to $T = 60$, and track L^1 errors over time.

Long Time Test L^1 Error - Alternating Flux, SSPRK4 (Exact Traveling Wave Solution)

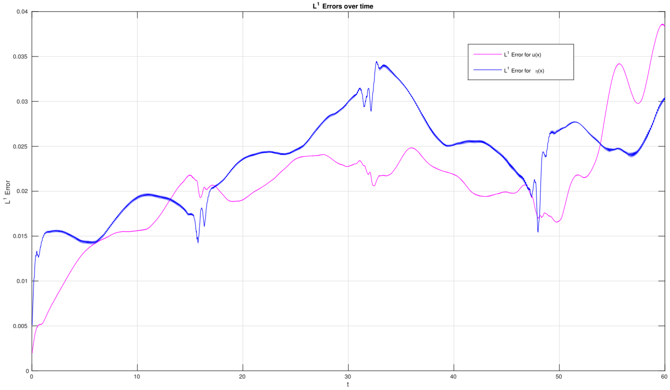

Figure : L^1 errors plotted against time.

Long Time Test Approximation - Alternating Flux, Midpoint in Time (Exact Traveling Wave Solution)

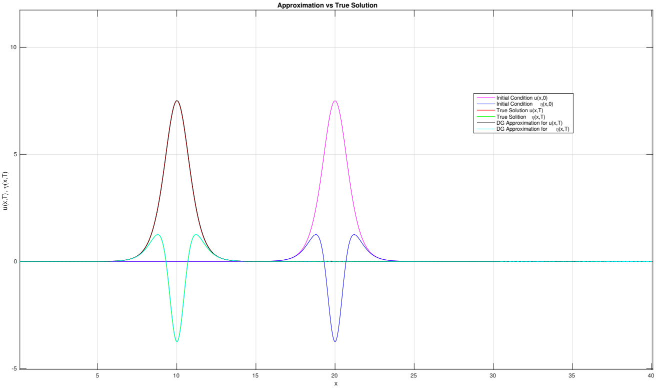

Figure : For the long time test, we run the code up to $T = 60$, and track L^1 errors over time.

Long Time Test L^1 Error - Alternating Flux, Midpoint in Time (Exact Traveling Wave Solution)

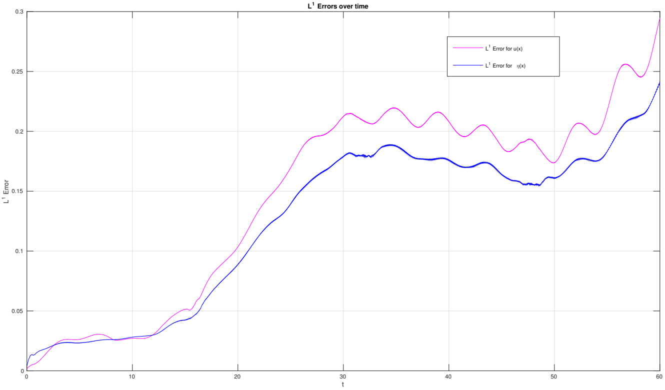

Figure : L^1 errors plotted against time.

Conserved Quantity - Alternating-SSPRK4-Midpoint Comparison (Exact Traveling Wave Solution)

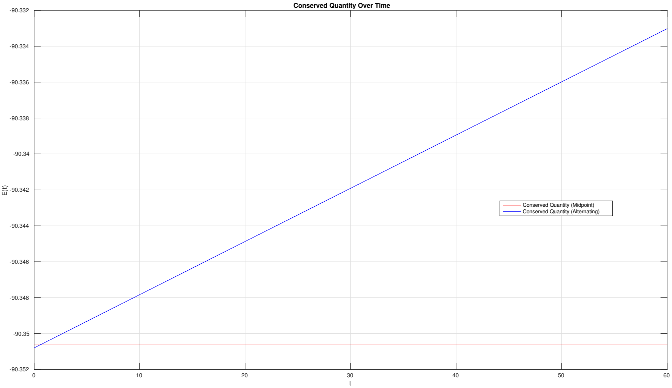

Figure : Comparison of Energy Values of SSPRK4 and Midpoint, with Alternating Flux.

Solitary Wave Generation Test

For the solitary wave generation test, we start with a first order approximation to the traveling wave solution that was used in the mesh refinement, and long time tests. The initial condition is given by

$$\eta(x,0) = \eta_0 \, \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{3\eta_0}{k}}(x - x_0)\right),$$
$$u(x,0) = \eta(x,0) - \frac{1}{4}\eta(x,0)^2,$$

where $\eta_0 = 0.8$ is the peak height for $\eta(x,0)$, and $x_0 = 20$.

The wave is evolved over the long domain, then “filtered”, and reset back to the left hand side of the domain. The process is repeated until dispersive tails are “small.”

Solitary Wave Generation Test Initial Condition

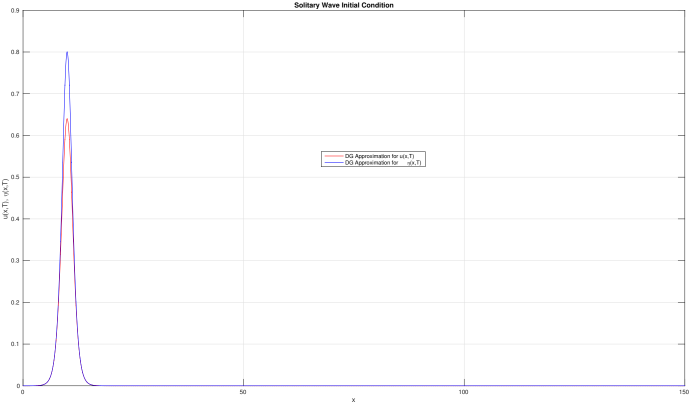

Figure : Solitary wave initial condition profile.

Solitary Wave Generation Test - One Evolution (T = 42)

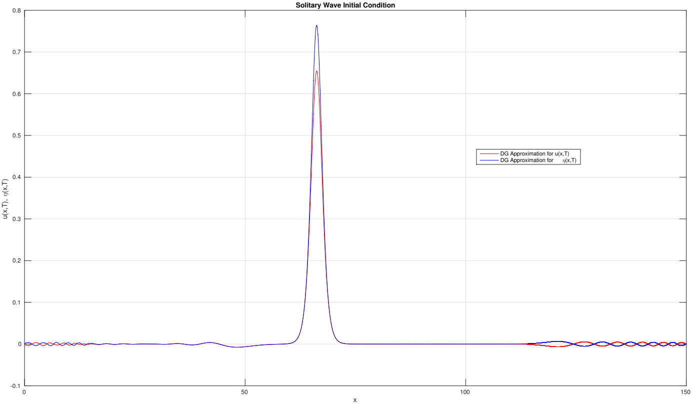

Figure : Solitary wave propagation at $T = 42$.

Solitary Wave Collision Test

Solitary Wave Collision Test

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Concluding Remarks

- Local Discontinuous Galerkin (LDG) method to solve the single BBM equation and BBM-system.
- Alternating and upwind flux choices that conserve energy and work well for long time simulations.
- Stability results and error estimates for the proposed method.
- Numerical experiments validating the usefulness of the method.
- Error estimate proofs for the single BBM and coupled BBM-system are still in progress.

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Thank you for your attention!