1. Algebra of Functions and Function Composition

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Examples based on Section 2.2 and 2.3 of College Algebra by Beecher

Given two functions \( f(x) \) and \( g(x) \) (or even more than two!) we can create new functions by doing basic manipulations or an operation called composition.

<table>
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<th>Definition: Sum, Difference, Product, and Quotient of Functions</th>
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<td>If ( f ) and ( g ) are functions of ( x ) then we have the following new functions:</td>
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<td>(1) The <strong>sum</strong> ( f + g ) is defined by ( (f + g)(x) = f(x) + g(x) ) where ( x ) is in the domain of both ( f ) and ( g ). So the domain of ( f + g ) is the intersection of the domain of ( f ) and the domain of ( g ).</td>
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<td>(2) The <strong>difference</strong> ( f - g ) is defined by ( (f - g)(x) = f(x) - g(x) ) where ( x ) is in the domain of both ( f ) and ( g ). So the domain of ( f - g ) is the intersection of the domain of ( f ) and the domain of ( g ).</td>
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<td>(3) The <strong>product</strong> ( fg ) is defined by ( (fg)(x) = f(x)g(x) ) where ( x ) is in the domain of both ( f ) and ( g ). So the domain of ( fg ) is the intersection of the domain of ( f ) and the domain of ( g ).</td>
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<td>(4) The <strong>quotient</strong> ( f/g ) is defined by ( (f/g)(x) = f(x)/g(x) ) where ( x ) is in the domain of both ( f ) and ( g ) and ( g(x) \neq 0 ). So the domain of ( f/g ) is the intersection of the domain of ( f ) and the domain of ( g ), followed by removing any points where ( g(x) = 0 ).</td>
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**Example 1.1**

Given \( f(x) = x + 4 \) and \( g(x) = \sqrt{x - 1} \), find \( f + g, fg, g/f, f/g \), simplify the expressions, if possible, and identify the domain of each function.

**Solution.** First it is helpful to identify the domain of \( f \) and \( g \) first. We have Domain\(_f\) = \( \mathbb{R} = (-\infty, \infty) \) and Domain\(_g\) = \( [1, \infty) \). The domain of \( g \) is \( [1, \infty) \) because the square root function cannot have negative inputs, so the input \( x - 1 \) is negative (that is \( x - 1 < 0 \)) whenever \( x < 1 \).

Now \( (f + g)(x) = x + 4 + \sqrt{x - 1} \) is the new function and the domain will be Domain\(_{f+g}\) = Domain\(_f\) \( \cap \) Domain\(_g\) = \( [1, \infty) \). It is recommended that you plot the domains on a number line to see why the new domain is \( [1, \infty) \).

Similarly, \( (fg)(x) = (x + 4)\sqrt{x - 1} = x\sqrt{x - 1} + 4\sqrt{4} \) is the new function and the domain will be Domain\(_{fg}\) = Domain\(_f\) \( \cap \) Domain\(_g\) = \( [1, \infty) \).

Next, \( (g/f)(x) = \frac{\sqrt{x - 1}}{x + 4} \) is the new function. To find the domain we first calculate the intersection of the domains: Domain\(_f\) \( \cap \) Domain\(_g\) = \( [1, \infty) \). Now we need to remove any points where \( f(x) = 0 \), which happens when \( x + 4 = 0 \) or \( x = -4 \). In this case, this point is not included in \( [1, \infty) \), so Domain\(_{g/f}\) = \( [1, \infty) \).

Finally, \( (f/g)(x) = \frac{x + 4}{\sqrt{x - 1}} = \frac{x}{\sqrt{x - 1}} + \frac{4}{\sqrt{x - 1}} \) is the new function. To find the domain we first calculate the intersection of the domains: Domain\(_f\) \( \cap \) Domain\(_g\) = \( [1, \infty) \). Now we need to remove any points where \( g(x) = 0 \), which happens when \( \sqrt{x - 1} = 0 \) or \( x = 1 \). Hence Domain\(_{f/g}\) = \( (1, \infty) \).

**Example 1.2**

Given \( f(x) = x - 3 \) and \( g(x) = \sqrt{x + 4} \), find \( g/f, f/g \), simplify the expressions, if possible, and identify the domain of each function.

**Solution.** Again, identify the domain of \( f \) and \( g \) first. We have Domain\(_f\) = \( \mathbb{R} = (-\infty, \infty) \) and Domain\(_g\) = \( [-4, \infty) \).

Now, \( (g/f)(x) = \frac{\sqrt{x + 4}}{x - 3} \) is the new function. To find the domain we first calculate the intersection of the domains: Domain\(_f\) \( \cap \) Domain\(_g\) = \( [-4, \infty) \). Now we need to remove any points where \( f(x) = 0 \), which happens when \( x - 3 = 0 \) or \( x = 3 \). So Domain\(_{g/f}\) = \( [-4, 3) \cup (3, \infty) \).

Finally, \( (f/g)(x) = \frac{x - 3}{\sqrt{x + 4}} = \frac{x}{\sqrt{x + 4}} - \frac{3}{\sqrt{x + 4}} \) is the new function. To find the domain we first calculate the intersection of the domains: Domain\(_f\) \( \cap \) Domain\(_g\) = \( [-4, \infty) \). Now we need to remove any points where \( g(x) = 0 \), which happens when \( \sqrt{x + 4} = 0 \) or \( x = -4 \). Hence Domain\(_{f/g}\) = \( (-4, \infty) \).
Now we look at the last form: composition.

**Definition: Composition of Functions**

If \( f \) and \( g \) are functions of \( x \) then we have the following new function:

1. The **composition** \( f \circ g \) defined by \((f \circ g)(x) = f(g(x))\) where \( x \) is in the domain of \( g \) and \( g(x) \) is in the domain of \( f \).

2. The **composition** \( g \circ f \) defined by \((g \circ f)(x) = g(f(x))\) where \( x \) is in the domain of \( f \) and \( f(x) \) is in the domain of \( g \).

**Example 1.3**

Given \( f(x) = x^2 + x - 5 \), \( g(x) = x + 3 \), \( h(x) = x^3 \), calculate \( f \circ g \), \( g \circ f \), \( f \circ h \) and find the domains.

**Solutions.** In doing function composition, first thing you should do is find the domain of the 'inside' function. In these cases, \( f, g, h \) are polynomials so their domain is \( \mathbb{R} \). Next we calculate the compositions.

First, \((f \circ g)(x) = f(g(x)) = f(x + 3) = (x + 3)^2 + (x + 3) - 5 = x^2 + 6x + 9 + x + 3 - 5 = x^2 + 7x + 7.\) This is also a polynomial, so the domain is \( \mathbb{R} \) for \( f \circ g \).

Second, \((g \circ f)(x) = g(f(x)) = g(x^2 + x - 5) = (x^2 + x - 5) + 3 = x^2 + x - 2.\) This is also a polynomial, so the domain is \( \mathbb{R} \) for \( g \circ f \).

Finally, \((f \circ h)(x) = f(h(x)) = f(x^3) = (x^3)^2 + (x^3) - 5 = x^6 + x^3 - 5.\) This is also a polynomial, so the domain is \( \mathbb{R} \) for \( f \circ h \).

**Example 1.4**

Given \( f(x) = x^2 \), \( g(x) = \sqrt{x+4} \), \( h(x) = \frac{1}{x-2} \), calculate \( f \circ g \), \( g \circ f \), \( g \circ h \) and find the domains.

**Solutions.** Again, first thing you should do is find the domain of the 'inside' function.

First, for \((f \circ g)(x) = f(g(x))\), the inside function is \( g(x) \) and the domain of \( g(x) \) is \([-4, \infty)\). Next \( f(g(x)) = f(\sqrt{x+4}) = (\sqrt{x+4})^2 = x + 4 \). This is a polynomial, so the domain is \( \mathbb{R} \), but since it came from a function composition, we restrict our domain to the domain of the input function \( g \). Hence the domain of \( f \circ g \) is \([-4, \infty)\).

Second, for \((g \circ f)(x) = g(f(x))\), the inside function is \( f(x) \) and the domain of \( f(x) \) is \( \mathbb{R} \). Next \( g(f(x)) = g(x^2) = \sqrt{x^2 + 4} \). This is a square root function, so we need to make what is inside of the radical is not negative. However, \( x^2 + 4 \) is always positive. Hence the domain of \( g \circ f \) is \( \mathbb{R} \).

Finally, for \((g \circ h)(x) = g(h(x))\), the inside function is \( h(x) \) and the domain of \( h(x) \) is \((-\infty, 2) \cup (2, \infty)\) or all real numbers except 2. Next \( g(h(x)) = g \left( \frac{1}{x-2} \right) = \sqrt{\frac{1}{x-2} + 4} = \sqrt{\frac{4x-7}{x-2}} \). This is a square root function, so again whatever is inside of radical sign cannot be negative, so we would need to figure out when \( \frac{4x-7}{x-2} < 0 \). This function is actually a rational function, so in a later chapter you will techniques for how to determine the sign of this expression. Since this is not the point of this example, I will tell you that \( \frac{4x-7}{x-2} < 0 \) whenever \( \frac{7}{2} < x < 2 \). Hence when we consider both restrictions, \((-\infty, 2) \cup (2, \infty)\) and \( \frac{7}{2} < x < 2 \), the domain will be \( \frac{7}{2} < x < 2 \).