1. Piecewise Functions
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We can create more complicated functions by considering **Piece-wise functions**.

**Definition: Piecewise-function.**
A *piecewise-function* is a function defined by multiple functions, where each function’s domain is defined by some rule (such as $x \in (-1, 1)$) that may be disjoint from the other functions’ rule.

$$f(x) = \begin{cases} g(x), & \text{if } x \in A \\ h(x), & \text{if } x \in B \end{cases}$$

**Example 1.1**
The function

$$f(x) = \begin{cases} x, & \text{if } x \in (-3, 0) \\ x^2, & \text{if } x \in [0, 3) \end{cases}$$

is a piecewise function whose domain is the union of each sub-function’s interval, that is $(-3, 3)$. The functions are $x$ and $x^2$. If $x \in (-3, 0)$, then $f(x) = x$, while if $x \in [0, 3)$ then $f(x) = x^2$.

For example, $f(-2) = -2$ while $f(0) = 0^2 = 0$ and $f(.5) = (.5)^2 = .25$.

Evaluating piecewise functions is relatively easy compared to **graphing** them, so let’s describe a method for graphing them.

However, it is very important that before we describe the method, I urge you to go back to your notes and review how to graph basic functions first! If you don’t know how to graph lines, polynomials, the absolute value function, square root functions, and constant functions, then graphing piecewise functions will be hard. In addition, it is important to understand the meaning of something like $(2, \infty)$ or $x > 2$. Review interval notation and inequality notation.

**Steps: Graphing a piecewise function**

Say we have a piecewise function given by

$$f(x) = \begin{cases} g(x), & \text{if } x \in A \\ h(x), & \text{if } x \in B \end{cases}$$

Then do the following in the order described.

1. Look at the first function, $g(x)$ and see what is its rule, $A$. Typically $A$ will be an interval or written as an inequality. There are several type of intervals we can have.
   - (a) Closed interval, such as $[a, b]$ or $a \leq x \leq b$ in inequality form.
   - (b) Open interval, such as $(a, b)$ and $(a, \infty)$ or $a < x < b$ and $x > a$ in inequality form.
   - (c) Half-open interval, such as $(a, b]$ and $(-\infty, a]$ or $a < x \leq b$ and $x \leq a$ in inequality form.

2. If it is a closed interval, evaluate $g$ at those end points, say $x = a$ and $x = b$ and plot those points $(a, g(a))$ and $(b, g(b))$ on a graph using filled in dots. Filled in dots indicate they are included in the graph.

3. If it is an open interval with finite end points (that is something like $(a, b)$), evaluate $g$ at those end points and plot $(a, g(a))$ and $(b, g(b))$ on a graph using open circles. Open circles indicate they are not included in the graph.

4. If it is an open interval with one finite end point (say something like $a > x$ or $(a, \infty)$), evaluate $g$ at the finite end point and plot that point $(a, g(a))$ on a graph using open circles. To deal with the $\infty$ or $-\infty$, you will need to draw an arrow to indicate the function continues on.

5. If it is a half-open interval, do the same thing as before which is evaluate the function on the finite end points and use filled in dots or open circles depending on if the end point is included or not and use arrows to indicate the function continues on.

6. Once you take care of endpoints, you draw the rest of the function and connect the points by drawing the general shape of the function. If $g(x)$, then connect the points by line. If $g(x) = x^2$, then connect the points by a curve.

7. Do the same analysis for $g(x)$!
Example 1.2

Graph

\[ f(x) = \begin{cases} 
  x, & \text{if } -3 < x < 0 \\
  x^2, & \text{if } 0 \leq x < 3
\end{cases} \]

Solution. The first function is \( g(x) = x \) whose rule is \(-3 < x < 0\). We evaluate at the end points to obtain the points \((-3, -3)\) and \((0, 0)\). Now we graph them with open circles.

\[ \begin{array}{c}
  \text{Next, we draw the points in between. Since } g(x) = x, \text{ we draw a straight line connecting the points.} \\
  \end{array} \]

This finishes the first function.

Now the next function is \( h(x) = x^2 \) whose rule is \( 0 \leq x < 3 \). We evaluate at the end points to obtain \((0, 0)\) and \((3, 9)\). Now we plot \((0, 0)\) with a filled in dot since \( x = 0 \) is included in \( 0 \leq x < 3 \) while \((3, 9)\) is plotted with an open circle since \( x = 3 \) is not included in \( 0 \leq x < 3 \).

Now we fill in the space in between the points. Since \( h(x) = x^2 \), this is going to curve as follows.
Thus, we have graphed $f(x)$.

Example 1.3

Graph

$$f(x) = \begin{cases} 
3, & \text{if } x < 0 \\
\frac{1}{2}x + 4, & \text{if } 0 \leq x \leq 4 
\end{cases}$$

Solution. The first function is $g(x) = 3$ whose rule is $x < 0$ or $(-\infty, 0)$. We evaluate at the end point $x = 0$ to obtain $g(0) = 3$, so we graph $(0, 3)$. on the graph

Now since the rule is $x < 0$, we will need to draw an arrow to indicate we are continuing the graph indefinitely. This is where we recommend we plot more points, say $x = -1$, $x = -2$. Then we have $(-1, 3)$ and $(-2, 3)$ so we have

This finishes the first function.
Now the next function is \( h(x) = \frac{1}{2}x + 4 \) whose rule is \( 0 \leq x \leq 4 \). We evaluate at the end points to obtain \((0, 4)\) and \((4, 6)\). We plot both of these points with filed circles since \( x = 0 \) and \( x = 4 \) are included in \( 0 \leq x \leq 4 \).

Now we fill in the space in between the points. Since \( h(x) \) is a line, we connect the points with a straight line.

Thus, we have graphed \( f(x) \).

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**Example 1.4**

Graph

\[
f(x) = \begin{cases} 
-4x + 2, & \text{if } -2 < x < 0 \\
5x - 1, & \text{if } 0 \leq x \leq 5.
\end{cases}
\]

**Solution.** The first function is \( g(x) = -4x + 2 \) whose rule is \(-2 < x < 0\). We evaluate at the end points to obtain \((-2, 10)\) and \((0, 2)\), which we leave as open circles.
Now since \( g(x) \) is a line, we connect the points with a line.

This finishes the first function.

Now the next function is \( h(x) = 5x - 1 \) whose rule is \( 0 \leq x \leq 5 \). We evaluate at the end points to obtain \((0, -1)\) and \((5, 9)\). We plot both of these points with filled circles since \( x = 0 \) and \( x = 5 \) are included in \( 0 \leq x \leq 5 \).

Now we fill in the space in between the points. Since \( h(x) \) is a line, we connect the points with a straight line.
Thus, we have graphed $f(x)$. 