1. Symmetry
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Examples based on Section 2.4 of College Algebra by Beecher

Certain functions’ graphs have symmetry properties. First, we give examples of graphs of these functions

**Definition: Symmetric with respect to x-axis**

These functions satisfy the property that if we fold the graph on the x-axis, the parts above and below the x-axis will coincide.

**Example 1.1: x-axis**

The function $x = y^2$.

![Graph of $x = y^2$]

**Definition: Symmetric with respect to y-axis**

These functions satisfy the property that if we fold the graph on the y-axis, the parts to the left and right of the y-axis will coincide.
Example 1.2: $x$-axis

The function $y^2 = x$.

![Graph of $y^2 = x$](image1)

Definition: Symmetric with respect to Origin

These functions satisfy the property that if we rotate the graph $180^\circ$ about the origin, the resulting figure coincides with the original.

Example 1.3: $x$-axis

The function $x^2 = y^2 + 2$.

![Graph of $x^2 = y^2 + 2$](image2)

We can test symmetry without looking at the graph of these functions:
Definition: Checking Symmetry Without a graph

Given a function $y = f(x)$

1. **Symmetric with respect to the $x$-axis:** Replace $y$ by $-y$ and check if we still obtain the original function $f(x)$.

2. **Symmetric with respect to the $y$-axis:** Replace $x$ by $-x$ and check if we still obtain the original function $f(x)$.

3. **Symmetric with respect to the origin:** Replace $x$ by $-x$ and $y$ by $-y$ and check if we still obtain the original function $f(x)$.

Example 1.4

Determine the symmetry of $3x = |y|$.

**Proof.** We check symmetric with respect to the $x$-axis. Replace $y$ by $-y$:

\[
3x = |-y| \quad 3x = |y|.
\]

Notice we obtain our original function again! So this is symmetric with respect to the $x$-axis.

Now we check symmetric with respect to the $y$-axis. Replace $x$ by $-x$:

\[
3(-x) = |y| \quad -3x = |y|.
\]

Notice this is not the same function as $3x = |y|$ so we do not have symmetry with respect to the $y$-axis.

Finally we check symmetry with respect to the origin. Replace $x$ by $-x$ and $y$ by $-y$:

\[
3(-x) = |-y| \quad -3x = |y|.
\]

We do not get the same function as before so it is not symmetric with respect to the origin.

These symmetries are related to even and odd functions.

Definition: Even and Odd Functions

Given a function $y = f(x)$,

1. **Even function or symmetric with respect to $y$-axis:** Replace $x$ by $-x$ and check if $f(x) = f(-x)$

2. **Odd function or symmetric respect to $x$-axis:** Replace $x$ by $-x$ and check if $f(-x) = -f(x)$. A function cannot be both even and odd, unless it is the zero function.

Example 1.5

Determine the symmetry of if $f(x) = \sqrt{x^2 + 1}$, $g(x) = x + \frac{1}{x}$ and $h(x) = 7x^3 + 4x - 2$ are even, odd, or neither.

**Proof.** To check even, odd, or neither, all we need to do is calculate $f(-x)$, $g(-x)$ and $h(-x)$:

\[
f(-x) = \sqrt{(-x)^2 + 1} = \sqrt{x^2 + 1},
\]

\[
g(-x) = (-x) + \frac{1}{-x} = -x - \frac{1}{x},
\]

\[
h(-x) = 7(-x)^3 + 4(-x) - 2 = -7x^3 - 4x - 2.
\]

Since $f(-x) = f(x)$, $f$ is even.

We first see that $g(-x) \neq g(x)$, but we can factor out a negative from both terms to obtain $g(-x) = -x - \frac{1}{x} = -(x + \frac{1}{x}) = -g(x)$. Hence $g$ is odd.

However, $h(-x) \neq h(x)$ so $h$ is not even. If we factor a negative out from every term, we have $h(-x) = -7x^3 - 4x - 2 = -(7x^3 + 4x - 2)$ which does not equal $-h(x) = -(7x^3 + 4x - 2)$. So $h$ is neither even nor odd.