1. Solving Rational and Radical Expressions
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Examples based on Section 3.4 of College Algebra by Beecher

2. Rational Expressions

We know how to solve \( \frac{5}{x} + 1 = 3 \) and \( x^2 + 4x + 6 = 0 \), but what about something like

\[
\frac{t + 1}{3} - \frac{t - 1}{2} = 1
\]

or

\[
\frac{2}{x^2 - 9} + \frac{5}{x - 3} = \frac{3}{x + 3}.
\]

We will learn how. First a definition.

**Definition: Rational Expressions**

A **rational expression** is of the form

\[ R(x) = \frac{P(x)}{Q(x)} \]

where \( P, Q \) are polynomials.

Now for how to solve.

**Steps: Solving Rational Expressions**

To solve an equality involving Rational Expressions say

\[
\frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} = \frac{P_3(x)}{Q_3(x)}
\]

multiply the whole equality by the common denominator so that the \( Q_1(x), Q_2(x), Q_3(x) \) in the denominators cancel out and you get a new expressions \( A, B, C \) and a new equality

\[ A(x) + B(x) = C(x) \]

that will be easier to solve for.

**Example 2.1**

Solve

\[
\frac{t + 1}{3} - \frac{t - 1}{2} = 1.
\]

**Solution.** Multiply the equality by the common denominator \( 3 \cdot 2 = 6 \) to get

\[
\frac{t + 1}{3} - \frac{t - 1}{2} = 1
\]

\[
6 \left( \frac{t + 1}{3} - \frac{t - 1}{2} \right) = 6 \cdot 1
\]

\[
2(t + 1) - 3(t - 1) = 6
\]

\[
2t + 2 - 3t + 3 = 6
\]

\[
-t + 5 = 6
\]

\[
t = -1
\]

We plug it back in and get that the equality holds. ■

**Example 2.2**

Solve

\[
\frac{2}{x^2 - 9} + \frac{5}{x - 3} = \frac{3}{x + 3}.
\]
Solution. Our denominators are $x^2 - 9 = (x-3)(x+3)$, $x-3$ and $x+3$, so the common denominator is $(x+3)(x-3)$. Hence

$$\frac{2}{x^2 - 9} + \frac{5}{x - 3} = \frac{3}{x + 3}$$

$$(x + 3)(x - 3) \left( \frac{2}{x^2 - 9} + \frac{5}{x - 3} \right) = (x + 3)(x - 3) \frac{3}{x + 3}$$

$$2 + 5(x + 3) = 3(x - 3)$$

$$5x + 17 = 3x - 9$$

$$x = -13.$$

We plug it back in and get that the equality holds.

3. Radical Expressions

Now how do we solve something like $\sqrt{x - 4} + 1 = 5$? Because of the radical, we cannot solve for $x$ directly since $\sqrt{x - 5} \neq \sqrt{x} - \sqrt{5}$. First we give a definition then we explain how.

**Definition: Radical Expressions**

A **radical expression** is of the form

$$R(x) = \sqrt{P(x)}, R(x) = \sqrt[n]{Q(x)}$$

where $P, Q$ are polynomials.

Now for how to solve.

**Steps: Solving Radical Expressions**

1. To solve an equality involving Radical Expressions say

$$\sqrt{P(x)} + Q(x) = R(x)$$

First move the radical expression to one side and move everything else to the other side.

$$\sqrt{P(x)} = R(x) - Q(x)$$

Now $n = 2$, square both sides to get ride of the square root. If $n = 3$, cube both sides to get ride of the cube root. And similarly for any $n$-th power. So we get

$$P(x) = (R(x) - Q(x))^n$$

and we just expand the right side out by foiling and the resulting expression should be easier to solve for.

2. To solve an equality like

$$\sqrt{P(x)} + \sqrt{Q(x)} = R(x)$$

that is where we have two radical expressions, first isolate one of the radical expressions and move the other one to the other side.

$$\sqrt{P(x)} = R(x) - \sqrt{Q(x)}$$

Now as before raise both sides to the 2-th power.

$$P(x) = (R(x) - \sqrt{Q(x)})^2$$

And when we expand the right side we get

$$P(x) = R(x)^2 - 2R(x)\sqrt{Q(x)} + Q(x)$$

Now we only have one radical expression. From here we do the same as before and move all the non-radical expressions to one side. See (1) to proceed from here.

**IMPORTANT** Whatever solutions you end up getting, you must check your work and see if when you plug them back in your expressions are defined. There should not be any negatives in the radical!

**Example 3.1**

Solve

$$\sqrt{x - 4} + 1 = 5.$$
Solution.

\[
\sqrt{x-4} + 1 = 5 \\
\sqrt{x-4} = 4 \\
(\sqrt{x-4})^2 = 4^2 \\
x - 4 = 16 \\
x = 20
\]

Plugging it back in we get \( \sqrt{20-4} + 1 = \sqrt{16} + 1 = 4 + 1 = 5 \) as desired.

Example 3.2

Solve

\[
\sqrt{2x+1} - \sqrt{x} = 1.
\]

Solution. This involves two radical expressions, so pick one expression to leave on one side and move the other one to the other side.

\[
\sqrt{2x+1} - \sqrt{x} = 1 \\
\sqrt{2x+1} = 1 + \sqrt{x} \\
(\sqrt{2x+1})^2 = (1 + \sqrt{x})^2 \\
2x + 1 = (1 + \sqrt{x})(1 + \sqrt{x}) \\
2x + 1 = 1 + \sqrt{x} + \sqrt{x} + x \\
2x + 1 = 1 + 2\sqrt{x} + x \\
2x + 1 - 1 - x = 2\sqrt{x} \\
x = 2\sqrt{x} \\
x^2 = 4x \\
x^2 - 4x = 0 \\
x(x - 4) = 0.
\]

So we get \( x = 0 \) and \( x = 4 \). We plug both back in and see that everything is defined and we get back 1. Hence \( x = 0 \) and \( x = 4 \) are our solutions.