

COURANT ALGEBROIDS FROM CATEGORIFIED SYMPLECTIC GEOMETRY

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ABSTRACT. In categorified symplectic geometry, one studies the categorified algebraic and geometric structures that naturally arise on manifolds equipped with a closed nondegenerate $(n + 1)$ -form. The case relevant to classical string theory is when $n = 2$ and is called ‘2-plectic geometry’. Just as the Poisson bracket makes the smooth functions on a symplectic manifold into a Lie algebra, there is a Lie 2-algebra of observables associated to any 2-plectic manifold. String theory, closed 3-forms and Lie 2-algebras also play important roles in the theory of Courant algebroids. Courant algebroids are vector bundles which generalize the structures found in tangent bundles and quadratic Lie algebras. It is known that a particular kind of Courant algebroid (called an exact Courant algebroid) naturally arises in string theory, and that such an algebroid is classified up to isomorphism by a closed 3-form on the base space, which then induces a Lie 2-algebra structure on the space of global sections. In this paper we begin to establish precise connections between 2-plectic manifolds and Courant algebroids. We prove that any manifold M equipped with a 2-plectic form ω gives an exact Courant algebroid E_ω over M with Ševera class $[\omega]$, and we construct an embedding of the Lie 2-algebra of observables into the Lie 2-algebra of sections of E_ω . We then show that this embedding identifies the observables as particular infinitesimal symmetries of E_ω which preserve the 2-plectic structure on M .

1. INTRODUCTION

The underlying geometric structures of interest in categorified symplectic geometry are multisymplectic manifolds: manifolds equipped with a closed, nondegenerate form of degree ≥ 2 [8]. This kind of geometry originated in the work of DeDonder [10] and Weyl [24] on the calculus of variations, and more recently has been used as a formalism to investigate classical field theories [11, 12, 13]. In this paper, we call a manifold ‘ n -plectic’ if it is equipped with a closed nondegenerate $(n + 1)$ -form. Hence ordinary symplectic geometry corresponds to the $n = 1$ case, and the corresponding 1-dimensional field theory is just the classical mechanics of point particles. In general, examples of n -plectic manifolds include phase spaces suitable for describing n -dimensional classical field theories. We will be primarily concerned with

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the $n = 2$ case. This is the first really new case of n -plectic geometry and the corresponding 2-dimensional field theories of interest include bosonic string theory. Indeed, just as the phase space of the classical particle is a manifold equipped with a closed, nondegenerate 2-form, the phase space of the classical string is a *finite-dimensional* manifold equipped with a closed nondegenerate 3-form. This phase space is often called the ‘multiphase space’ of the string [11] in order to distinguish it from the infinite-dimensional symplectic manifolds that are used as phase spaces in string field theory [6].

In classical mechanics, the relevant mathematical structures are not just geometric, but also algebraic. The symplectic form gives the space of smooth functions the structure of Poisson algebra. Analogously, in classical string theory, the 2-plectic form induces a bilinear skew-symmetric bracket on a particular subspace of differential 1-forms, which we call Hamiltonian. The Hamiltonian 1-forms and smooth functions form the underlying chain complex of an algebraic structure known as a semistrict Lie 2-algebra. A semistrict Lie 2-algebra can be viewed as a categorified Lie algebra in which the Jacobi identity is weakened and is required to hold only up to isomorphism. Equivalently, it can be described as a 2-term L_∞ -algebra, i.e. a generalization of a 2-term differential graded Lie algebra in which the Jacobi identity is only satisfied up to chain homotopy [1, 14]. Just as the Poisson algebra of smooth functions represents the observables of a system of particles, it has been shown that the Lie 2-algebra of Hamiltonian 1-forms contains the observables of the classical string [2]. In general, an n -plectic structure will give rise to a L_∞ -algebra on an n -term chain complex of differential forms in which the $(n - 1)$ -forms correspond to the observables of an n -dimensional classical field theory [15].

Many of the ingredients found in 2-plectic geometry are also found in the theory of Courant algebroids, which was also developed by generalizing structures found in symplectic geometry. Courant algebroids were first used by Courant [9] to study generalizations of pre-symplectic and Poisson structures in the theory of constrained mechanical systems. Roughly, a Courant algebroid is a vector bundle that generalizes the structure of a tangent bundle equipped with a symmetric nondegenerate bilinear form on the fibers. In particular, the underlying vector bundle of a Courant algebroid comes equipped with a skew-symmetric bracket on the space of global sections. However, unlike the Lie bracket of vector fields, the bracket need not satisfy the Jacobi identity.

In a letter to Weinstein, Ševera [20] described how a certain type of Courant algebroid, known as an exact Courant algebroid, appears naturally when studying 2-dimensional variational problems. In classical string theory, the string can be represented as a map $\phi: \Sigma \rightarrow M$ from a 2-dimensional parameter space Σ into a manifold M corresponding to space-time. The image $\phi(\Sigma)$ is called the string world-sheet. The map ϕ extremizes the integral of a 2-form $\theta \in \Omega^2(M)$ over its world-sheet. Hence the classical string is a solution to a 2-dimensional variational problem. The 2-form θ is called

the Lagrangian and depends on elements of the first jet bundle of the trivial bundle $\Sigma \times M$. The Lagrangian is not unique. A solution ϕ remains invariant if an exact 1-form or ‘divergence’ is added to θ . It is, in fact, the 3-form $d\theta$ that is relevant. In this context, Ševera observed that the 3-form $d\theta$ uniquely specifies (up to isomorphism) the structure of an exact Courant algebroid over M . The general correspondence between exact Courant algebroids and closed 3-forms on the base space was further developed by Ševera, and also by Bressler and Chervov [4], to give a complete classification. An exact Courant algebroid over M is determined up to isomorphism by its Ševera class: an element $[\omega]$ in the third de Rham cohomology of M .

Just as in 2-plectic geometry, the underlying geometric structure of a Courant algebroid has an algebraic manifestation. Roytenberg and Weinstein [16] showed that the bracket on the space of global sections induces an L_∞ structure. If we are considering an exact Courant algebroid, then the global sections can be identified with ordered pairs of vector fields and 1-forms on the base space. Roytenberg and Weinstein’s results imply that these sections, when combined with the smooth functions on the base space, form a semistrict Lie 2-algebra [23]. Moreover, the bracket of the Lie 2-algebra is determined by a closed 3-form corresponding to a representative of the Ševera class [21].

Thus there are striking similarities between 2-plectic manifolds and exact Courant algebroids. Both originate from attempts to generalize certain aspects of symplectic geometry. Both come equipped with a closed 3-form that gives rise to a Lie 2-algebra structure on a chain complex consisting of smooth functions and differential 1-forms. In this paper, we prove that there is indeed a connection between the two. We show that any manifold M equipped with a 2-plectic form ω gives an exact Courant algebroid E_ω with Ševera class $[\omega]$, and that there is an embedding of the Lie 2-algebra of observables into the Lie 2-algebra corresponding to E_ω . Moreover, this embedding allows us to characterize the Hamiltonian 1-forms as particular infinitesimal symmetries of E_ω which preserve the 2-plectic structure on M .

2. COURANT ALGEBROIDS

Here we recall some basic facts and examples of Courant algebroids and then we proceed to describe Ševera’s classification of exact Courant algebroids. There are several equivalent definitions of a Courant algebroid found in the literature. In this paper we use the definition given by Roytenberg [17].

Definition 2.1. *A Courant algebroid is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a skew-symmetric bracket $[[\cdot, \cdot]]$ on $\Gamma(E)$, and a bundle map (called the **anchor**) $\rho: E \rightarrow TM$ such that for all $e_1, e_2, e_3 \in \Gamma(E)$ and for all $f, g \in C^\infty(M)$ the following properties hold:*

$$(1) \quad [[e_1, [[e_2, e_3]]] - [[[[e_1, e_2], e_3]] - [e_2, [[e_1, e_3]]] = -\mathcal{D}T(e_1, e_2, e_3),$$

- (2) $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)],$
- (3) $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2 - \frac{1}{2}\langle e_1, e_2 \rangle \mathcal{D}f,$
- (4) $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0,$
- (5) $\rho(e_1) (\langle e_2, e_3 \rangle) = \langle [e_1, e_2] + \frac{1}{2}\mathcal{D}\langle e_1, e_2 \rangle, e_3 \rangle + \langle e_2, [e_1, e_3] + \frac{1}{2}\mathcal{D}\langle e_1, e_3 \rangle \rangle,$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, $\mathcal{D}: C^\infty(M) \rightarrow \Gamma(E)$ is the map defined by $\langle \mathcal{D}f, e \rangle = \rho(e)f$, and

$$T(e_1, e_2, e_3) = \frac{1}{6} (\langle [[e_1, e_2], e_3] + [[e_3, e_1], e_2] + [[e_2, e_3], e_1] \rangle).$$

The bracket in Definition 2.1 is skew-symmetric, but the first property implies that it needs only to satisfy the Jacobi identity “up to $\mathcal{D}T$ ”. (The notation suggests we think of this as a boundary.) The function T is often referred to as the **Jacobiator**. (When there is no risk of confusion, we shall refer to the Courant algebroid with underlying vector bundle $E \rightarrow M$ as E .)

Note that the vector bundle E may be identified with E^* via the bilinear form $\langle \cdot, \cdot \rangle$ and therefore we have the dual map

$$\rho^*: T^*M \rightarrow E.$$

Hence the map \mathcal{D} is simply the pullback of the de Rham differential by ρ^* .

There is an alternate definition given by Ševera [20] for Courant algebroids which uses a bilinear operation on sections that satisfies a Jacobi identity but is not skew-symmetric.

Definition 2.2. A **Courant algebroid** is a vector bundle $E \rightarrow M$ together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a bilinear operation \circ on $\Gamma(E)$, and a bundle map $\rho: E \rightarrow TM$ such that for all $e_1, e_2, e_3 \in \Gamma(E)$ and for all $f \in C^\infty(M)$ the following properties hold:

- (1) $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3),$
- (2) $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)],$
- (3) $e_1 \circ fe_2 = f(e_1 \circ e_2) + \rho(e_1)(f)e_2,$
- (4) $e_1 \circ e_1 = \frac{1}{2}\mathcal{D}\langle e_1, e_1 \rangle,$
- (5) $\rho(e_1) (\langle e_2, e_3 \rangle) = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle,$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, and $\mathcal{D}: C^\infty(M) \rightarrow \Gamma(E)$ is the map defined by $\langle \mathcal{D}f, e \rangle = \rho(e)f$.

The “bracket” \circ is related to the bracket given in Definition 2.1 by:

$$x \circ y = [[x, y] + \frac{1}{2}\mathcal{D}\langle x, y \rangle]. \quad (1)$$

Roytenberg [17] showed that if E is a Courant algebroid in the sense of Definition 2.1 with bracket $[[\cdot, \cdot]]$, bilinear form $\langle \cdot, \cdot \rangle$ and anchor ρ , then E is a Courant algebroid in the sense of Definition 2.2 with the same anchor and bilinear form but with bracket \circ given by Eq. 1. Unless otherwise stated, all Courant algebroids mentioned in this paper are Courant algebroids in the sense of Definition 2.1. We introduced Definition 2.2 mainly to connect our results here with previous results in the literature.

Example 1. An important example of a Courant algebroid is the **standard Courant algebroid** $E_0 = TM \oplus T^*M$ over any manifold M with bracket

$$\llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_0 = \left([v_1, v_2], \mathcal{L}_{v_1}\alpha_2 - \mathcal{L}_{v_2}\alpha_1 - \frac{1}{2}d(\iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1) \right), \quad (2)$$

and bilinear form

$$\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1. \quad (3)$$

In this case the anchor $\rho: E_0 \rightarrow TM$ is the projection map, and for a function $f \in C^\infty(M)$, $\mathcal{D}f = (0, df)$.

The standard Courant algebroid is the prototypical example of an **exact Courant algebroid** [4].

Definition 2.3. A Courant algebroid $E \rightarrow M$ with anchor map $\rho: E \rightarrow TM$ is **exact** iff

$$0 \rightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \rightarrow 0$$

is an exact sequence of vector bundles.

2.1. The Ševera class of an exact Courant algebroid. Ševera's classification originates in the idea that choosing a splitting of the above short exact sequence corresponds to defining a kind of connection.

Definition 2.4. A **connection** on an exact Courant algebroid E over a manifold M is a map of vector bundles $A: TM \rightarrow E$ such that

- (1) $\rho \circ A = \text{id}_{TM}$,
- (2) $\langle A(v_1), A(v_2) \rangle = 0$ for all $v_1, v_2 \in TM$,

where $\rho: E \rightarrow TM$ and $\langle \cdot, \cdot \rangle$ are the anchor and bilinear form, respectively.

If A is a connection and $\theta \in \Omega^2(M)$ is a 2-form then one can construct a new connection:

$$(A + \theta)(v) = A(v) + \rho^*\theta(v, \cdot). \quad (4)$$

$(A + \theta)$ satisfies the first condition of Definition 2.4 since $\ker \rho = \text{im } \rho^*$. The second condition follows from the fact that we have by definition of ρ^* :

$$\langle \rho^*(\alpha), e \rangle = \alpha(\rho(e)) \quad (5)$$

for all $e \in \Gamma(E)$ and $\alpha \in \Omega^1(M)$. Furthermore, one can show that any two connections on an exact Courant algebroid must differ (as in Eq. 4) by a 2-form on M . Hence the space of connections on an exact Courant algebroid is an affine space modeled on the vector space of 2-forms $\Omega^2(M)$ [4].

The failure of a connection to preserve the bracket gives a suitable notion of curvature:

Definition 2.5. If E is an exact Courant algebroid over M with bracket $\llbracket \cdot, \cdot \rrbracket$ and $A: TM \rightarrow E$ is a connection then the **curvature** is a map $F: TM \times TM \rightarrow E$ defined by

$$F(v_1, v_2) = \llbracket A(v_1), A(v_2) \rrbracket - A([v_1, v_2]).$$

If F is the curvature of a connection A then given $v_1, v_2 \in TM$, it follows from exactness and axiom 2 in Definition 2.1 that there exists a 1-form $\alpha_{v_1, v_2} \in \Omega^1(M)$ such that $F(v_1, v_2) = \rho^*(\alpha_{v_1, v_2})$. Since A is a connection, its image is isotropic in E . Therefore for any $v_3 \in TM$ we have:

$$\langle F(v_1, v_2), A(v_3) \rangle = \langle \llbracket A(v_1), A(v_2) \rrbracket, A(v_3) \rangle.$$

The above formula allows one to associate the curvature F to a 3-form on M :

Proposition 2.6. *Let E be an exact Courant algebroid over a manifold M with bracket $\llbracket \cdot, \cdot \rrbracket$ and bilinear form $\langle \cdot, \cdot \rangle$. Let $A: TM \rightarrow E$ be a connection on E . Then given vector fields v_1, v_2, v_3 on M :*

(1) *The function*

$$\omega(v_1, v_2, v_3) = \langle \llbracket A(v_1), A(v_2) \rrbracket, A(v_3) \rangle$$

defines a closed 3-form on M .

(2) *If $\theta \in \Omega^2(M)$ is a 2-form and $\tilde{A} = A + \theta$ then*

$$\begin{aligned} \tilde{\omega}(v_1, v_2, v_3) &= \langle \llbracket \tilde{A}(v_1), \tilde{A}(v_2) \rrbracket, \tilde{A}(v_3) \rangle \\ &= \omega(v_1, v_2, v_3) + d\theta(v_1, v_2, v_3). \end{aligned}$$

Proof. The statements in the proposition are proven in Lemmas 4.2.6, 4.2.7, and 4.3.4 in the paper by Bressler and Chervov [4]. In their work they define a Courant algebroid using Definition 2.2, and therefore their bracket satisfies the Jacobi identity, but is not skew-symmetric. In our notation, their definition of the curvature 3-form is:

$$\omega'(v_1, v_2, v_3) = \langle A(v_1) \circ A(v_2), A(v_3) \rangle.$$

In particular they show that \circ satisfying the Jacobi identity implies ω' is closed. The Jacobiator corresponding to the Courant bracket is non-trivial in general. However the isotropicity of the connection and Eq. 1 imply

$$A(v_1) \circ A(v_2) = \llbracket A(v_1), A(v_2) \rrbracket \quad \forall v_1, v_2 \in TM.$$

Hence $\omega' = \omega$, so all the needed results in [4] apply here. \square

Thus the above proposition implies that the curvature 3-form of an exact Courant algebroid over M gives a well-defined cohomology class in $H_{\text{DR}}^3(M)$, independent of the choice of connection.

2.2. Twisting the Courant bracket. The previous section describes how to go from exact Courant algebroids to closed 3-forms. Now we describe the reverse process. In Example 1 we showed that one can define the standard Courant algebroid E_0 over any manifold M . The total space is the direct sum $TM \oplus T^*M$, the bracket and bilinear form are given in Eqs. 2 and 3, and the anchor is simply the projection. The inclusion $A(v) = (v, 0)$ of the tangent bundle into the direct sum is obviously a connection on E_0 and it is easy to see that the standard Courant algebroid has zero curvature.

Ševera and Weinstein [20, 21] observed that the bracket on E_0 could be twisted by a closed 3-form $\omega \in \Omega^3(M)$ on the base:

$$\llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_\omega = \llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_0 + \omega(v_1, v_2, \cdot).$$

This gives a new Courant algebroid E_ω with the same anchor and bilinear form. Using Eqs. 2 and 3 we can compute the curvature 3-form of this new Courant algebroid:

$$\begin{aligned} \langle \llbracket A(v_1), A(v_2) \rrbracket, A(v_3) \rangle &= \langle \llbracket (v_1, 0), (v_2, 0) \rrbracket, (v_3, 0) \rangle \\ &= \langle ([v_1, v_2], \omega(v_1, v_2, \cdot)), (v_3, 0) \rangle \\ &= \omega(v_1, v_2, v_3), \end{aligned}$$

and we see that E_ω is an exact Courant algebroid over M with Ševera class $[\omega]$.

3. 2-PECTIC GEOMETRY

We now give a brief overview of 2-plectic geometry. More details including motivation for several of the definitions presented here can be found in our previous work with Baez and Hoffnung [2, 3].

Definition 3.1. *A 3-form ω on a C^∞ manifold M is **2-plectic**, or more specifically a **2-plectic structure**, if it is both closed:*

$$d\omega = 0,$$

and nondegenerate:

$$\forall v \in T_x M, \iota_v \omega = 0 \Rightarrow v = 0$$

If ω is a 2-plectic form on M we call the pair (M, ω) a **2-plectic manifold**.

The 2-plectic structure induces an injective map from the space of vector fields on M to the space of 2-forms on M . This leads us to the following definition:

Definition 3.2. *Let (M, ω) be a 2-plectic manifold. A 1-form α on M is **Hamiltonian** if there exists a vector field v_α on M such that*

$$d\alpha = -\iota_{v_\alpha} \omega.$$

We say v_α is the **Hamiltonian vector field** corresponding to α . The set of Hamiltonian 1-forms and the set of Hamiltonian vector fields on a 2-plectic manifold are both vector spaces and are denoted as $\text{Ham}(M)$ and $\text{Vect}_H(M)$, respectively.

The Hamiltonian vector field v_α is unique if it exists, but note there may be 1-forms α having no Hamiltonian vector field. Furthermore, two distinct Hamiltonian 1-forms may differ by a closed 1-form and therefore share the same Hamiltonian vector field.

We can generalize the Poisson bracket of functions in symplectic geometry by defining a bracket of Hamiltonian 1-forms.

Definition 3.3. Given $\alpha, \beta \in \text{Ham}(M)$, the **bracket** $\{\alpha, \beta\}$ is the 1-form given by

$$\{\alpha, \beta\} = \iota_{v_\beta} \iota_{v_\alpha} \omega.$$

Proposition 3.4. Let $\alpha, \beta, \gamma \in \text{Ham}(M)$ and let $v_\alpha, v_\beta, v_\gamma$ be the respective Hamiltonian vector fields. The bracket $\{\cdot, \cdot\}$ has the following properties:

(1) The bracket of Hamiltonian forms is Hamiltonian:

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]} \omega, \quad (6)$$

so in particular we have

$$v_{\{\alpha, \beta\}} = [v_\alpha, v_\beta].$$

(2) The bracket is skew-symmetric:

$$\{\alpha, \beta\} = -\{\beta, \alpha\} \quad (7)$$

(3) The bracket satisfies the Jacobi identity up to an exact 1-form:

$$\{\alpha, \{\beta, \gamma\}\} - \{\{\alpha, \beta\}, \gamma\} - \{\beta, \{\alpha, \gamma\}\} = dJ_{\alpha, \beta, \gamma} \quad (8)$$

with $J_{\alpha, \beta, \gamma} = \iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega$.

Proof. See Proposition 3.7 in [2]. \square

4. LIE 2-ALGEBRAS

Both the Courant bracket and the bracket on Hamiltonian 1-forms are, roughly, Lie brackets which satisfy the Jacobi identity up to an exact 1-form. This leads us to the notion of a Lie 2-algebra: a category equipped with structures analogous to those of a Lie algebra, for which the usual laws involving skew-symmetry and the Jacobi identity hold up to isomorphism [1, 19]. A Lie 2-algebra in which the isomorphisms are actual equalities is called a strict Lie 2-algebra. A Lie 2-algebra in which the laws governing skew-symmetry are equalities but the Jacobi identity holds only up to isomorphism is called a semistrict Lie 2-algebra.

Here we define a semistrict Lie 2-algebra to be a 2-term chain complex of vector spaces equipped with structures analogous to those of a Lie algebra, for which the usual laws hold up to chain homotopy. In this guise, a semistrict Lie 2-algebra is nothing more than a 2-term L_∞ -algebra. For more details, we refer the reader to the work of Lada and Stasheff [14], and the work of Baez and Crans [1].

Definition 4.1. A **semistrict Lie 2-algebra** is a 2-term chain complex of vector spaces $L = (L_1 \xrightarrow{d} L_0)$ equipped with:

- a chain map $[\cdot, \cdot]: L \otimes L \rightarrow L$ called the **bracket**;
- an antisymmetric chain homotopy $J: L \otimes L \otimes L \rightarrow L$ from the chain map

$$\begin{aligned} L \otimes L \otimes L &\rightarrow L \\ x \otimes y \otimes z &\mapsto [x, [y, z]], \end{aligned}$$

to the chain map

$$\begin{aligned} L \otimes L \otimes L &\rightarrow L \\ x \otimes y \otimes z &\mapsto [[x, y], z] + [y, [x, z]] \end{aligned}$$

called the **Jacobiator**,

such that the following equation holds:

$$\begin{aligned} [x, J(y, z, w)] + J(x, [y, z], w) + J(x, z, [y, w]) + [J(x, y, z), w] \\ + [z, J(x, y, w)] = J(x, y, [z, w]) + J([x, y], z, w) \\ + [y, J(x, z, w)] + J(y, [x, z], w) + J(y, z, [x, w]). \end{aligned} \quad (9)$$

We will also need a suitable notion of morphism:

Definition 4.2. *Given semistrict Lie 2-algebras L and L' with bracket and Jacobiator $[\cdot, \cdot]$, J and $[\cdot, \cdot]'$, J' respectively, a **homomorphism** from L to L' consists of:*

- a chain map $\phi = (\phi_0, \phi_1) : L \rightarrow L'$, and
- a chain homotopy $\phi_2 : L \otimes L \rightarrow L$ from the chain map

$$\begin{aligned} L \otimes L &\rightarrow L \\ x \otimes y &\mapsto [\phi(x), \phi(y)]', \end{aligned}$$

to the chain map

$$\begin{aligned} L \otimes L &\rightarrow L \\ x \otimes y &\mapsto \phi([x, y]) \end{aligned}$$

such that the following equation holds:

$$\begin{aligned} J'(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(J(x, y, z)) = \\ \phi_2(x, [y, z]) - \phi_2([x, y], z) - \phi_2(y, [x, z]) - [\phi_2(x, y), \phi_0(z)]' \\ + [\phi_0(x), \phi_2(y, z)]' - [\phi_0(y), \phi_2(x, z)]'. \end{aligned} \quad (10)$$

This definition is equivalent to the definition of a morphism between 2-term L_∞ -algebras. (The same definition is given in [1], but it contains a typographical error.)

4.1. The Lie 2-algebra from a 2-plectic manifold. Given a 2-plectic manifold (M, ω) , we can construct a semistrict Lie 2-algebra. The underlying 2-term chain complex is namely:

$$L = C^\infty(M) \xrightarrow{d} \text{Ham}(M)$$

where d is the usual exterior derivative of functions. This chain complex is well-defined, since any exact form is Hamiltonian, with 0 as its Hamiltonian vector field. We can construct a chain map

$$\{\cdot, \cdot\} : L \otimes L \rightarrow L,$$

by extending the bracket on $\text{Ham}(M)$ trivially to L . In other words, in degree 0, the chain map is given as in Definition 3.3:

$$\{\alpha, \beta\} = \iota_{v_\beta} \iota_{v_\alpha} \omega,$$

and in degrees 1 and 2, we set it equal to zero:

$$\{\alpha, f\} = 0, \quad \{f, \alpha\} = 0, \quad \{f, g\} = 0.$$

The precise construction of this Lie 2-algebra is given in the following theorem:

Theorem 4.3. *If (M, ω) is a 2-plectic manifold, there is a semistrict Lie 2-algebra $L(M, \omega)$ where:*

- the space of 0-chains is $\text{Ham}(M)$,
- the space of 1-chains is C^∞ ,
- the differential is the exterior derivative $d: C^\infty \rightarrow \text{Ham}(M)$,
- the bracket is $\{\cdot, \cdot\}$,
- the Jacobiator is the linear map $J_L: \text{Ham}(M) \otimes \text{Ham}(M) \otimes \text{Ham}(M) \rightarrow C^\infty$ defined by $J_L(\alpha, \beta, \gamma) = \iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega$.

Proof. See Theorem 4.4 in [2]. □

4.2. The Lie 2-algebra from a Courant algebroid. Given any Courant algebroid $E \rightarrow M$ with bilinear form $\langle \cdot, \cdot \rangle$, bracket $\llbracket \cdot, \cdot \rrbracket$, and anchor $\rho: E \rightarrow TM$, we can construct a 2-term chain complex

$$C = C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E),$$

with differential $\mathcal{D} = \rho^*d$. The bracket $\llbracket \cdot, \cdot \rrbracket$ on global sections can be extended to a chain map $\llbracket \cdot, \cdot \rrbracket: C \otimes C \rightarrow C$. If e_1, e_2 are degree 0 chains then $\llbracket e_1, e_2 \rrbracket$ is the original bracket. If e is a degree 0 chain and f, g are degree 1 chains, then we define:

$$\begin{aligned} \llbracket e, f \rrbracket &= -\llbracket f, e \rrbracket = \frac{1}{2} \langle e, \mathcal{D}f \rangle \\ \llbracket f, g \rrbracket &= 0. \end{aligned}$$

This extended bracket gives a semistrict Lie 2-algebra on the complex C :

Theorem 4.4. *If E is a Courant algebroid, there is a semistrict Lie 2-algebra $C(E)$ where:*

- the space of 0-chains is $\Gamma(E)$,
- the space of 1-chains is $C^\infty(M)$,
- the differential the map $\mathcal{D}: C^\infty(M) \rightarrow \Gamma(M)$,
- the bracket is $\llbracket \cdot, \cdot \rrbracket$,
- the Jacobiator is the linear map $J_C: \Gamma(M) \otimes \Gamma(M) \otimes \Gamma(M) \rightarrow C^\infty(M)$ defined by

$$\begin{aligned} J_C(e_1, e_2, e_3) &= -T(e_1, e_2, e_3) \\ &= -\frac{1}{6} (\langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + \langle \llbracket e_3, e_1 \rrbracket, e_2 \rangle + \langle \llbracket e_2, e_3 \rrbracket, e_1 \rangle). \end{aligned}$$

Proof. The proof that a Courant algebroid in the sense of Definition 2.1 gives rise to a semistrict Lie 2-algebra follows from the work done by Roytenberg on graded symplectic supermanifolds [18] and Lie 2-algebras [19]. In particular we refer the reader to Example 5.4 of [19] and Section 4 of [18].

On the other hand, the original construction of Roytenberg and Weinstein [16] gives a L_∞ -algebra on the complex:

$$0 \rightarrow \ker \mathcal{D} \xrightarrow{\iota} C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E),$$

with trivial structure maps l_n for $n \geq 3$. Moreover, the map l_2 (corresponding to the bracket $\llbracket \cdot, \cdot \rrbracket$ given above) is trivial in degree > 1 and the map l_3 (corresponding to the Jacobiator J_C) is trivial in degree > 0 . Hence we can restrict this L_∞ -algebra to our complex C and use the results in [1] that relate L_∞ -algebras with semistrict Lie 2-algebras. \square

5. THE COURANT ALGEBROID ASSOCIATED TO A 2-PLECTIC MANIFOLD

Now we have the necessary machinery in place to describe how Courant algebroids connect with 2-plectic geometry. First, recall the discussion in Section 2.2 on twisting the bracket of the standard Courant algebroid E_0 by a closed 3-form. From Definition 3.1, we immediately have the following:

Proposition 5.1. *Let (M, ω) be a 2-plectic manifold. There exists an exact Courant algebroid E_ω with Ševera class $[\omega]$ over M with underlying vector bundle $TM \oplus T^*M \rightarrow M$, anchor $\rho(v, \alpha) = v$, and bracket and bilinear form given by:*

$$\begin{aligned} \llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_\omega &= \left([v_1, v_2], \mathcal{L}_{v_1} \alpha_2 - \mathcal{L}_{v_2} \alpha_1 - \frac{1}{2} d(\iota_{v_1} \alpha_2 - \iota_{v_2} \alpha_1) + \iota_{v_2} \iota_{v_1} \omega \right), \\ \langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle &= \iota_{v_1} \alpha_2 + \iota_{v_2} \alpha_1. \end{aligned}$$

More importantly, the Courant algebroid constructed in Proposition 5.1 not only encodes the 2-plectic structure ω , but also the corresponding Lie 2-algebra constructed in Theorem 4.3:

Theorem 5.2. *Let (M, ω) be a 2-plectic manifold and let E_ω be its corresponding Courant algebroid. Let $L(M, \omega)$ and $C(E_\omega)$ be the semistrict Lie 2-algebras corresponding to (M, ω) and E_ω , respectively. Then there exists a homomorphism embedding $L(M, \omega)$ into $C(E_\omega)$.*

Before we prove the theorem, we introduce some lemmas to ease the calculations. In the notation that follows, if α, β are Hamiltonian 1-forms with corresponding vector fields v_α, v_β , then

$$B(\alpha, \beta) = \frac{1}{2}(\iota_{v_\alpha} \beta - \iota_{v_\beta} \alpha). \quad (11)$$

Also by the symbol \circlearrowleft we mean cyclic permutations of the symbols α, β, γ .

Lemma 5.3. *If $\alpha, \beta \in \text{Ham}(M)$ with corresponding Hamiltonian vector fields v_α, v_β , then $\mathcal{L}_{v_\alpha} \beta = \{\alpha, \beta\} + dt_{v_\alpha} \beta$.*

Proof. Since $\mathcal{L}_v = \iota_v d + dt_v$,

$$\mathcal{L}_{v_\alpha} \beta = \iota_{v_\alpha} d\beta + dt_{v_\alpha} \beta = -\iota_{v_\alpha} \iota_{v_\beta} \omega + dt_{v_\alpha} \beta = \{\alpha, \beta\} + dt_{v_\alpha} \beta. \quad \square$$

Lemma 5.4. *If $\alpha, \beta, \gamma \in \text{Ham}(M)$ with corresponding Hamiltonian vector fields $v_\alpha, v_\beta, v_\gamma$, then*

$$\iota_{[v_\alpha, v_\beta]}\gamma + \circlearrowleft = -3\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega + 2(\iota_{v_\alpha}dB(\beta, \gamma) + \iota_{v_\gamma}dB(\alpha, \beta) + \iota_{v_\beta}dB(\gamma, \alpha)).$$

Proof. The identity $\iota_{[v_1, v_2]} = \mathcal{L}_{v_1}\iota_{v_2} - \iota_{v_2}\mathcal{L}_{v_1}$ and Lemma 5.3 imply:

$$\begin{aligned} \iota_{[v_\alpha, v_\beta]}\gamma &= \mathcal{L}_{v_\alpha}\iota_{v_\beta}\gamma - \iota_{v_\beta}\mathcal{L}_{v_\alpha}\gamma \\ &= \mathcal{L}_{v_\alpha}\iota_{v_\beta}\gamma - \iota_{v_\beta}(\{\alpha, \gamma\} + d\iota_{v_\alpha}\gamma) \\ &= \iota_{v_\alpha}d\iota_{v_\beta}\gamma - \iota_{v_\beta}\iota_{v_\gamma}\iota_{v_\alpha}\omega - \iota_{v_\beta}d\iota_{v_\alpha}\gamma, \end{aligned}$$

where the last equality follows from the definition of the bracket.

Hence it follows that:

$$\begin{aligned} \iota_{[v_\gamma, v_\alpha]}\beta &= \iota_{v_\gamma}d\iota_{v_\alpha}\beta - \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega - \iota_{v_\alpha}d\iota_{v_\gamma}\beta, \\ \iota_{[v_\beta, v_\gamma]}\alpha &= \iota_{v_\beta}d\iota_{v_\gamma}\alpha - \iota_{v_\gamma}\iota_{v_\alpha}\iota_{v_\beta}\omega - \iota_{v_\gamma}d\iota_{v_\beta}\alpha. \end{aligned}$$

Finally, note $2\iota_{v_\alpha}dB(\beta, \gamma) = \iota_{v_\alpha}d\iota_{v_\beta}\gamma - \iota_{v_\alpha}d\iota_{v_\gamma}\beta$. □

Lemma 5.5. *If $\alpha, \beta \in \text{Ham}(M)$ with corresponding Hamiltonian vector fields v_α, v_β , then*

$$\mathcal{L}_{v_\beta}\alpha - \mathcal{L}_{v_\alpha}\beta = -2(\{\alpha, \beta\} + dB(\alpha, \beta)).$$

Proof. Follows immediately from Lemma 5.3 and the definition of $B(\alpha, \beta)$. □

Proof of Theorem 5.2. We will construct a homomorphism from $L(M, \omega)$ to $C(E_\omega)$. Let L be the underlying chain complex of $L(M, \omega)$ consisting of Hamiltonian 1-forms in degree 0 and smooth functions in degree 1. Let C be the underlying chain of $C(E_\omega)$ consisting of global sections of E_ω in degree 0 and smooth functions in degree 1. The bracket $[[\cdot, \cdot]]_\omega$ denotes the extension of the bracket on $\Gamma(E_\omega)$ to the complex C in the sense of Theorem 4.4. Let $\phi_0: L_0 \rightarrow C_0$ be given by

$$\phi_0(\alpha) = (v_\alpha, -\alpha),$$

where v_α is the Hamiltonian vector field corresponding to α . Let $\phi_1: L_1 \rightarrow C_1$ be given by

$$\phi_1(f) = -f.$$

Finally let $\phi_2: L_0 \otimes L_0 \rightarrow C_1$ be given by

$$\phi_2(\alpha, \beta) = -B(\alpha, \beta) = -\frac{1}{2}(\iota_{v_\alpha}\beta - \iota_{v_\beta}\alpha).$$

Now we show ϕ_2 is a well-defined chain homotopy in the sense of Definition 4.2. For degree 0 we have:

$$\begin{aligned} [[\phi_0(\alpha), \phi_0(\beta)]]_\omega &= \left([v_\alpha, v_\beta], \mathcal{L}_{v_\alpha}(-\beta) - \mathcal{L}_{v_\beta}(-\alpha) + \frac{1}{2}d(\iota_{v_\alpha}\beta - \iota_{v_\beta}\alpha) + \iota_{v_\beta}\iota_{v_\alpha}\omega \right) \\ &= ([v_\alpha, v_\beta], -\{\alpha, \beta\} + d\phi_2(\alpha, \beta)), \end{aligned}$$

where the last equality above follows from Lemma 5.5. By Proposition 3.4, the Hamiltonian vector field of $\{\alpha, \beta\}$ is $[v_\alpha, v_\beta]$. Hence we have:

$$\llbracket \phi_0(\alpha), \phi_0(\beta) \rrbracket_\omega - \phi_0(\{\alpha, \beta\}) = d\phi_2(\alpha, \beta).$$

In degree 1, the bracket $\{\cdot, \cdot\}$ is trivial. Hence it follows from the definition of $\llbracket \cdot, \cdot \rrbracket_\omega$ and the bilinear form on E_ω (given in Proposition 5.1) that

$$\llbracket \phi_0(\alpha), \phi_1(f) \rrbracket_\omega = -\llbracket \phi_1(f), \phi_0(\alpha) \rrbracket_\omega = \frac{1}{2} \langle (v_\alpha, -\alpha), (0, -df) \rangle = \phi_2(\alpha, df).$$

Therefore ϕ_2 is a chain homotopy.

It remains to show the coherence condition (Eq. 10 in Definition 4.2) is satisfied. We rewrite the Jacobiator J_C as:

$$\begin{aligned} J_C(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) &= -\frac{1}{6} \langle \llbracket \phi_0(\alpha), \phi_0(\beta) \rrbracket, \phi_0(\gamma) \rangle + \odot \\ &= -\frac{1}{6} \langle ([v_\alpha, v_\beta], -\{\alpha, \beta\} - dB(\alpha, \beta)), (v_\gamma, -\gamma) \rangle + \odot \\ &= \frac{1}{6} \left(\iota_{[v_\alpha, v_\beta]} \gamma + \iota_{v_\gamma} \iota_{v_\beta} \iota_{v_\alpha} \omega + \iota_{v_\gamma} dB(\alpha, \beta) \right) + \odot \\ &= -J_L(\alpha, \beta, \gamma) + \frac{1}{2} (\iota_{v_\gamma} dB(\alpha, \beta) + \odot). \end{aligned}$$

The last equality above follows from Lemma 5.4 and the definition of the Jacobiator J_L . Therefore the left-hand side of Eq. 10 is

$$J_C(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) - \phi_1(J_L(\alpha, \beta, \gamma)) = \frac{1}{2} (\iota_{v_\gamma} dB(\alpha, \beta) + \odot).$$

By the skew-symmetry of the brackets, the right-hand side of Eq. 10 can be rewritten as:

$$(\llbracket \phi_0(\alpha), \phi_2(\beta, \gamma) \rrbracket_\omega + \odot) - (\phi_2(\{\alpha, \beta\}, \gamma) + \odot).$$

From the definitions of the bracket, bilinear form and ϕ_2 we have:

$$\begin{aligned} \llbracket \phi_0(\alpha), \phi_2(\beta, \gamma) \rrbracket_\omega + \odot &= \frac{1}{2} \langle (v_\alpha, -\alpha), (0, d\phi_2(\beta, \gamma)) \rangle + \odot \\ &= -\frac{1}{2} \iota_{v_\alpha} dB(\beta, \gamma) + \odot, \end{aligned}$$

and:

$$\begin{aligned} \phi_2(\{\alpha, \beta\}, \gamma) + \odot &= -\frac{1}{2} \left(\iota_{[v_\alpha, v_\beta]} \gamma - \iota_{v_\gamma} \iota_{v_\beta} \iota_{v_\alpha} \omega \right) \\ &= -(\iota_{v_\alpha} dB(\beta, \gamma) + \odot). \end{aligned}$$

The last equality above follows again from Lemma 5.4. Therefore the right-hand side of Eq. 10 is

$$\frac{1}{2} (\iota_{v_\gamma} dB(\alpha, \beta) + \odot).$$

Hence the maps ϕ_0, ϕ_1, ϕ_2 give a homomorphism of semistrict Lie 2-algebras. \square

We note that Roytenberg [19] has shown that a Courant algebroid defined using Definition 2.2 with the bilinear operation \circ induces a hemistrict Lie 2-algebra on the complex C described in Theorem 4.4 above. A hemistrict Lie 2-algebra is a Lie 2-algebra in which the skew-symmetry holds up to isomorphism, while the Jacobi identity holds as an equality. We have proven in previous work [2] that a 2-plectic structure also gives rise to a hemistrict Lie 2-algebra on the complex described in Theorem 4.3. One can show that all results presented above, in particular Theorem 5.2, carry over to the hemistrict case.

6. HAMILTONIAN 1-FORMS AS INFINITESIMAL SYMMETRIES OF THE COURANT ALGEBROID

Given a 2-plectic manifold (M, ω) , the Lie 2-algebra of observables $L(M, \omega)$ identifies particular infinitesimal symmetries of the corresponding Courant algebroid E_ω via the embedding described in the proof of Theorem 5.2. To see this, we first recall some basic facts concerning automorphisms of exact Courant algebroids. The presentation here follows the work of Bursztyn, Cavalcanti, and Gualtieri [7].

Definition 6.1. *Let $E \rightarrow M$ be a Courant algebroid with bilinear form $\langle \cdot, \cdot \rangle$, bracket $\llbracket \cdot, \cdot \rrbracket$, and anchor $\rho: E \rightarrow TM$. An **automorphism** is a bundle isomorphism $F: E \rightarrow E$ covering a diffeomorphism $\varphi: M \rightarrow M$ such that*

- (1) $\varphi^* \langle F(e_1), F(e_2) \rangle = \langle e_1, e_2 \rangle$,
- (2) $F(\llbracket e_1, e_2 \rrbracket) = \llbracket F(e_1), F(e_2) \rrbracket$,
- (3) $\rho(F(e_1)) = \varphi_*(\rho(e_1))$.

Consider the exact Courant algebroid E_ω described in Section 2.2 with underlying vector bundle $TM \oplus T^*M \rightarrow M$ and Ševera class $[\omega] \in H_{\text{DR}}^3(M)$. Given a 2-form $B \in \Omega^2(M)$, one can define a bundle isomorphism

$$\exp B: TM \oplus T^*M \rightarrow TM \oplus T^*M$$

by

$$\exp B(v, \alpha) = (v, \alpha + \iota_v B).$$

The map $\exp B$ is known as a ‘gauge transformation’. It covers the identity $\text{id}: M \rightarrow M$ and therefore is compatible (in the sense of Definition 6.1) with the anchor $\rho(v, \alpha) = v$. Since B is skew-symmetric, $\exp B$ preserves the bilinear form $\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \iota_{v_1} \alpha_2 + \iota_{v_2} \alpha_1$. However a simple computation shows that $\exp B$ preserves the bracket $\llbracket \cdot, \cdot \rrbracket_\omega$ (defined in Eq. 2.2) if and only if B is a closed 2-form:

$$\llbracket \exp B(v_1, \alpha_1), \exp B(v_2, \alpha_2) \rrbracket_\omega = \exp B(\llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_{\omega + dB}).$$

Given a diffeomorphism $\varphi: M \rightarrow M$ of the base space, one can define a bundle isomorphism $\Phi: TM \oplus T^*M \rightarrow TM \oplus T^*M$ by

$$\Phi(v, \alpha) = \left(\varphi_* v, (\varphi^*)^{-1} \alpha \right).$$

The map Φ satisfies conditions 1 and 3 of Definition 6.1 but does not preserve the bracket in general:

$$\llbracket \Phi(v_1, \alpha_1), \Phi(v_2, \alpha_2) \rrbracket_\omega = \Phi \left(\llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_{\varphi^*\omega} \right).$$

Bursztyn, Cavalcanti, and Gualtieri [7] showed that any automorphism F of the exact Courant algebroid E_ω must be of the form

$$F = \Phi \exp B, \quad (12)$$

where Φ is constructed from a diffeomorphism $\varphi: M \rightarrow M$ such that

$$\omega - \varphi^*\omega = dB. \quad (13)$$

This classification of automorphisms allows one to classify the infinitesimal symmetries as well. Let

$$F_t = \Phi_t \exp tB = \left(\varphi_{t*} \exp tB, (\varphi_t^*)^{-1} \exp tB \right)$$

be a 1-parameter family of automorphisms of the Courant algebroid E_ω with $F_0 = \text{id}_{E_\omega}$. Let $u \in \text{Vect}(M)$ be the vector field that generates the flow φ_{-t} . Then differentiation gives:

$$\left. \frac{dF_t}{dt}(v, \alpha) \right|_{t=0} = ([u, v], \mathcal{L}_u \alpha + \iota_v B).$$

Since $\omega - \varphi_t^*\omega = tdB$, it follows that u and B must satisfy the equality:

$$\mathcal{L}_u \omega = dB. \quad (14)$$

These infinitesimal transformations are called **derivations** [7] of the Courant algebroid E_ω , since they correspond to linear first order differential operators which act as derivations of the non-skew-symmetric bracket:

$$(v_1, \alpha_1) \circ_\omega (v_2, \alpha_2) = \llbracket (v_1, \alpha_1), (v_2, \alpha_2) \rrbracket_\omega + \frac{1}{2} d\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle. \quad (15)$$

$$= ([v_1, v_2], \mathcal{L}_{v_1} \alpha_2 - \iota_{v_2} d\alpha_1 + \iota_{v_2} \iota_{v_1} \omega). \quad (16)$$

In general, derivations are pairs $(u, B) \in \text{Vect}(M) \oplus \Omega^2(M)$ satisfying Eq. 14. They act on global sections $(v, \alpha) \in \Gamma(E_\omega)$ by:

$$(u, B) \cdot (v, \alpha) = ([u, v], \mathcal{L}_u \alpha + \iota_v B).$$

Global sections themselves naturally act as derivations via an **adjoint action** [7]. Given $(u, \beta) \in \Gamma(E_\omega)$ let B be the 2-form

$$B = -d\beta + \iota_u \omega. \quad (17)$$

Define $\text{ad}_{(u, \beta)}: \Gamma(E_\omega) \rightarrow \Gamma(E_\omega)$ by

$$\text{ad}_{(u, \beta)}(v, \alpha) = (u, B) \cdot (v, \alpha) = ([u, v], \mathcal{L}_u \alpha + \iota_v (-d\beta + \iota_u \omega)). \quad (18)$$

One can see this is indeed the adjoint action in the usual sense if one considers the non-skew-symmetric bracket given in Eq. 15:

$$\text{ad}_{(u, \beta)}(v, \alpha) = (u, \beta) \circ_\omega (v, \alpha).$$

Recall that in the proof of Theorem 5.2 we constructed a homomorphism of Lie 2-algebras using the map $\phi_0: \text{Ham}(M) \rightarrow \Gamma(E_\omega)$ defined by

$$\phi_0(\alpha) = (v_\alpha, -\alpha),$$

where v_α is the Hamiltonian vector field corresponding to α . Comparing Eq. 17 to Definition 3.2 of a Hamiltonian 1-form, we see that a section $(u, \beta) \in \Gamma(E_\omega)$ is in the image of the map ϕ_0 if and only if its adjoint action $\text{ad}_{(u, \beta)}$ corresponds to the pair $(u, 0) \in \text{Vect}(M) \oplus \Omega^2(M)$. This implies that $\text{ad}_{(u, \beta)}$ preserves the 2-plectic structure on M and that $-\beta$ is a Hamiltonian 1-form with Hamiltonian vector field u . Also if u is complete, then Eqs. 12 and 13 imply that the 1-parameter family F_t of Courant algebroid automorphisms generated by $\text{ad}_{(u, \beta)}$ corresponds to a 1-parameter family of diffeomorphisms $\varphi_t: M \rightarrow M$ which preserve the 2-plectic structure:

$$\varphi_t^* \omega = \omega.$$

In analogy with symplectic geometry, we call such automorphisms **Hamiltonian 2-plectomorphisms**.

We provide the following proposition as a summary of the discussion presented in this section:

Proposition 6.2. *Let (M, ω) be a 2-plectic manifold and let E_ω be its corresponding Courant algebroid. There is a one-to-one correspondence between the Hamiltonian 1-forms $\text{Ham}(M)$ on (M, ω) and sections (u, β) of E_ω whose adjoint action satisfies the equality*

$$\text{ad}_{(u, \beta)}(v, \alpha) = (\mathcal{L}_u v, \mathcal{L}_u \alpha).$$

7. CONCLUSIONS

We suspect that the results presented here are preliminary and indicate a deeper relationship between 2-plectic geometry and the theory of Courant algebroids. For example, the discussion of connections and curvature in Section 2.1 is reminiscent of the theory of gerbes with connection [5], whose relationship with Courant algebroids has been already studied [4, 20]. In 2-plectic geometry, gerbes have been conjectured to play a role in the geometric quantization of a 2-plectic manifold [2]. It will be interesting to see how these different points of view complement each other.

In general, much work has been done on studying the geometric structures induced by Courant algebroids (e.g. Dirac structures, twisted Dirac structures). Perhaps this work can aid 2-plectic geometry since many geometric structures in this context are somewhat less understood or remain ill-defined (e.g. the notion of a 2-Lagrangian submanifold or 2-polarization).

On the other hand, n -plectic manifolds are well understood in the role they play in classical field theory [11], and are also understood algebraically in the sense that an n -plectic structure gives an n -term L_∞ -algebra on a chain complex of differential forms [15]. Perhaps these insights can aid in

understanding ‘higher’ analogs of Courant algebroids (e.g. Lie n -algebroids) and complement similar ideas discussed by Ševera in [22].

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