Statement of Research

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1 Introduction

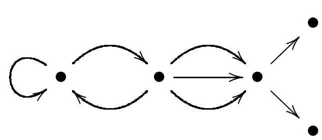
My current research lies in the fields of quantum algebra and representation theory. In particular, my recent work has studied the structure theory of quantum groups via Hall algebras. This work has also touched on aspects of category theory as a tool for gaining insight into the underlying details. In Section 2 I describe the motivation and basic ideas of Hall algebras. In Section 3 I describe several current results, and potential directions for continuing this work.

2 Hall Algebras

In the last few years my research has focused on the study of Hall algebras. Recently, Hall algebras have become of great interest because of their connection to quantum groups. It is known, due to Ringel [9], that the Hall algebra constructed from a quiver is isomorphic to 'half' of a certain quantum group related to the same quiver. This construction provides interesting insight into many structures on the quantum group.

We start by describing the construction of the Ringel-Hall algebra. We begin with a quiver Q (i.e. a directed graph) whose underlying graph is that of a simply-laced Dynkin diagram. We will then consider the abelian category $\text{Rep}(Q)$ of all finite dimensional representation of the quiver Q over a fixed finite field $\mathbb{F}_q$.

We start by fixing a finite field $\mathbb{F}_q$ and a directed graph D, which might look like this:



We shall call the category Q freely generated by D a **quiver**. The objects of Q are the vertices of D, while the morphisms are edge paths, with paths of length zero serving as identity morphisms.

By a **representation** of the quiver Q we mean a functor

$$R\colon Q \to \text{FinVect}_q,$$

where FinVect_q is the category of finite-dimensional vector spaces over $\mathbb{F}_q$. Such a representation simply assigns a vector space $R(d) \in \text{FinVect}_q$ to each vertex of D and a linear operator $R(e)\colon R(d) \to R(d')$ to each edge e from d to d'. By a **morphism** between representations of Q we mean a natural transformation between such functors. There is a category $\mathrm{Rep}(Q)$ where the objects are representations of Q and the morphisms are as above. This is an abelian category, so we can speak of indecomposable objects, short exact sequences, etc. in this category.

In 1972, Gabriel [4] discovered a remarkable fact. Namely: a quiver has finitely many isomorphism classes of indecomposable representations if and only if its underlying graph, ignoring the orientation of edges, is a finite disjoint union of Dynkin diagrams of type A, D or E. These are called **simply laced** Dynkin diagrams.

Henceforth, for simplicity, we assume the underlying graph of our quiver Q is a simply laced Dynkin diagram when we ignore the orientations of its edges. Let X be the underlying groupoid of $\mathrm{Rep}(Q)$: that is, the groupoid with representations of Q as objects and *isomorphisms* between these as morphisms. We will use this groupoid to construct the Hall algebra of Q.

As a vector space, the Hall algebra is just $\mathbb{R}[\underline{X}]$. Recall that this is the vector space whose basis consists of isomorphism classes of objects in X. In fancier language, it is the zeroth homology of X.

We now focus our attention on the Hall algebra product. Given three quiver representations M, N, and E, we define the set:

$$\mathcal{P}^E_{MN} = \{(f,g) : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \text{ is exact}\}$$

and we call its set cardinality P^E_{MN}. In the chosen category this set has a finite cardinality, since each representation is a finite-dimensional vector space over a finite field. The Hall algebra product counts these exact sequences, but with a subtle 'correction factor':

$$[M] \cdot [N] = \sum_{[E] \in \underline{X}} \frac{P^E_{MN}}{\mathrm{aut}(M)\,\mathrm{aut}(N)}\,[E]\,.$$

Where we call $\mathrm{aut}(M)$ the set cardinality of the group $\mathrm{Aut}(M)$.

Somewhat surprisingly, the above product is associative. In fact, Ringel [9] showed that the resulting algebra is isomorphic to the positive part $U_q^+\mathfrak{g}$ of the quantum group corresponding to our simply laced Dynkin diagram! So, roughly speaking, the Hall algebra of a simply laced quiver is 'half of a quantum group'.

This isomorphism also relates to a coalgebra structure on the Hall algebra. Using the same ideas from the multiplication formula, we can define a comultiplication on the Hall algebra to be a carefully weighted sum on ways to 'factor' a representation via short exact sequences. Formulaically this becomes:

$$\Delta(E) = \sum_{[M],[N] \in \underline{X}} \frac{P^E_{MN}}{\mathrm{aut}(E)}\,[N] \otimes [M]\,.$$

The logical next question to ask is whether these two maps satisfy the definition of a bialgebra (specifically is Δ multiplicative). Unfortunately the answer is 'no', but in a very interesting way.

3 Results and Future Work

3.1 A Hopf Algebra Structure on Hall Algebra

As we saw in the previous section, one problematic feature of Hall algebras is the fact that the standard multiplication and comultiplication maps do not satisfy the bialgebra compatibility condition in the underlying category Vect. In the past this problem has been resolved by working with a weaker structure called a 'twisted' bialgebra. In a recent paper of mine [13], I provide a different solution to the problem by first switching to the underlying category Vect^K of vector spaces graded by a group K called the Grothendieck group. We equip this category with a nontrivial braiding which depends on the K-grading. With this braiding, we find that the Hall algebra does satisfy the bialgebra condition exactly for the standard multiplication and comultiplication, and can also be equipped with an antipode, making it a Hopf algebra object in Vect^K.

3.2 A Categorification of Hall Algebras

My current short term goal for this project is to describe a categorification of Hall algebras through the groupoidification program.

Groupoidification is a new program of categorification first described by myself, Baez, and Hoffnung [3], from the ideas of Dolan and Trimble. The basic idea of groupoidification is to replace vector spaces with groupoids and linear operators with 'spans' of groupoids. This program has already successfully been applied to Hecke algebras by Hoffnung [5].

For our purposes, we can take clues from the construction methods for Hall algebras to see what the appropriate groupoids and spans should be to describe the structure of Hall algebras. Specifically, we recall that the underlying vector space of a Hall algebra is all finite linear combinations of equivalence class of quiver representations. We can thus choose the groupoid of such quiver representations as our replacement for the underlying vector space.

The next step is to define spans of groupoids which mimic the multiplication and comultiplication maps. First, set X equal to the underlying groupoid of $\mathrm{Rep}(Q)$. We start by defining a groupoid $\mathrm{SES}(Q)$ to serve as the apex of this span. An object of $\mathrm{SES}(Q)$ is a short exact sequence in $\mathrm{Rep}(Q)$, and a morphism between short exact sequences is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N' & \xrightarrow{f'} & E' & \xrightarrow{g'} & M' & \longrightarrow & 0 \end{array}$$

where α, β, and γ are isomorphisms of quiver representations.

Next, we define the span

$$X \xleftarrow{\;q\;} \mathrm{SES}(X) \xrightarrow{\;p\;} X \times X$$

where p is the projection of the quotient representation and subrepresentation, respectively, and q is projection of the extension. This span captures the idea behind the standard Hall algebra multiplication. Given two quiver representations M and N, this span relates them to every representation E that is an extension of M by N.

We can define a comultiplication on $\mathbb{R}[\underline{X}]$ using the the adjoint (i.e. mirror image) of the span that gives the multiplication.

In the same way that multiplication and comultiplication in standard Hall algebra did not form a bialgebra in Vect, these spans do not satisfy the bialgebra axiom in the 2-category Span(Grpd). However, we can again shift to a different kind of category, namely a braided monoidal 2-category. Through the details of this are still not complete, we have enough preliminary calculations done to suggest the following conjectures.

Conjecture 1. *The category* $\mathrm{Span}(\mathrm{Grpd} \downarrow X)$ *is a braided monoidal* 2*-category which 'groupoidifies'* Vect^K*.*

Conjecture 2. *With the multiplication and comultiplication spans above, the groupoidified Hall algebra* $X = \mathrm{Rep}(Q)_0$ *is a Hopf object in the braided monoidal* 2*-category* $\mathrm{Span}(\mathrm{Grpd} \downarrow X)$*.*

These conjectures together will provide us with a categorification of Hall algebras, and by extension a categorification of 'half' of a quantum group.

One potential direction I would like to take with this project is to begin studying the representation theory of Hall algebras through this same lens. Specifically, can we define the concept of a 'representation' of our groupoidified Hall algebra? If so, what features of standard representation theory can also be shifted up to this level?

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