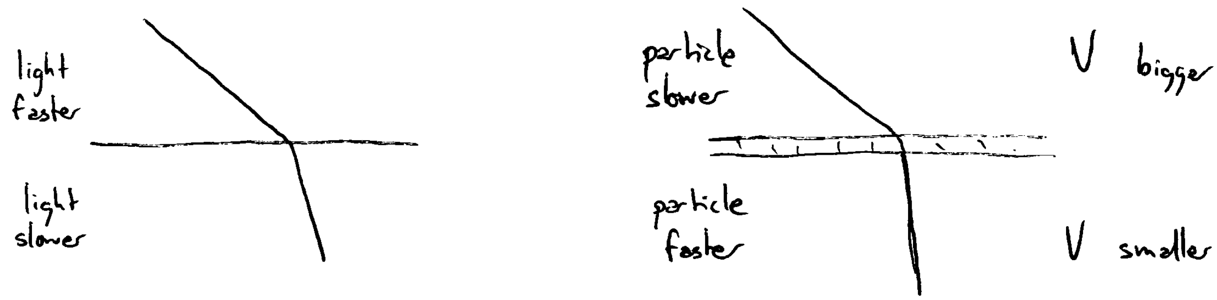
Least Action vs. Least Time

We mentioned Fermat's <u>principle of least time</u> in optics as analogous to the principle of least action in particle mechanics. This analogy is strange, since in the principle of least action we fix the time interval $q: [0,1] \to Q$. Also:



Nonetheless, Jacobi was able to reinterpret the mechanics of a particle in a potential as an optics problem & "unify" the two minimization principles. First let's consider light in a medium with a varying index of refraction n (recall $\frac{1}{n} \propto$ speed of light). Suppose it's in $\mathbb{R}^n$ with its usual Euclidean metric. If the light is trying to minimize the <u>time</u>, it's trying to minimize the arclength of its path in the metric

$$g_{ij} = n^2 \delta_{ij}$$

— i.e. index of refraction $n: \mathbb{R}^n \to (0,\infty)$ times the usual Euclidean metric

$$\delta_{ij} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

This is just like the free particle in general relativity (minimizing its proper time) except that now g_{ij} is a Riemannian metric:

$$g(v,w) = g_{ij} v^i w^j$$

w. $$g(v,v) \geq 0 .$$

So we'll use the same Lagrangian:

$$L(q,\dot{q}) = \sqrt{g_{ij}(q)\dot{q}^i\dot{q}^j}$$

& get the same Euler-Lagrange equations:

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0 \qquad ☆$$

if q is parameterized by arclength or more generally $\|\dot{q}\| = \sqrt{g_{ij}\dot{q}^i\dot{q}^j} = \text{constant}$. (where as before the Christoffel symbols Γ are built from derivatives of the metric g). A Now, what Jacobi did is show how the motion of a particle in a potential could be viewed as a special case of this. Consider a particle of mass m in Euclidean $\mathbb{R}^n$ with potential $V:\mathbb{R}^n \to \mathbb{R}$. It satisfies $F = ma$, i.e.

$$m\frac{d^2 q^i}{dt^2} = -\partial_i V \qquad ☆☆$$

How did Jacobi see ☆☆ as a special case of ☆? He considered a particle of energy E & he chose the index of refraction to be

$$n(q) = \sqrt{\frac{2}{m}(E - V(q))}$$

which is just the speed of a particle of energy E when the potential energy is $V(q)$, since

$$\sqrt{\frac{2}{m}(E-V)} = \sqrt{\frac{2}{m}K} = \sqrt{\frac{2}{m}\frac{1}{2}m\|\dot q\|^2} = \|\dot q\|.$$

Note: this is precisely backwards compared to optics, where $n(q)$ is proportional to the reciprocal of the speed of light!!! But let's see that it works:

$$\begin{aligned} L &= \sqrt{g_{ij}(q)\dot q^i \dot q^j} \\ &= \sqrt{n^2(q)\delta_{ij}\dot q^i \dot q^j} \\ &= \sqrt{\frac{2}{m}(E-V(q))\dot q^2} \end{aligned}$$

where $\dot q^2 = \dot q \cdot \dot q$ is just the usual Euclidean dot product $v \cdot w = \delta_{ij} v^i w^j$.

We get E-L equations:

$$p_i = \frac{\partial L}{\partial \dot q^i} = \sqrt{\frac{2}{m}(E-V)}\,\frac{\dot q_i}{\|\dot q\|}$$

$$\begin{aligned} F_i = \frac{\partial L}{\partial q^i} &= \partial_i \sqrt{\frac{2}{m}(E-V(q))}\ \ \|\dot q\| \\ &= \frac{1}{2}\,\frac{-\frac{2}{m}\partial_i V}{\sqrt{\frac{2}{m}(E-V)}}\,\|\dot q\| \end{aligned}$$

$\dot p = F$ says:

$$\frac{d}{dt}\sqrt{\frac{2}{m}(E-V)}\,\frac{\dot q_i}{\|\dot q\|} = -\frac{1}{m}\partial_i V\,\frac{\|\dot q\|}{\sqrt{\frac{2}{m}(E-V)}}$$

Jacobi noticed that this is just $F = ma$, or $m\ddot q_i = -\partial_i V$, if we reparameterize q so that $\|\dot{q}\| = \sqrt{\frac{2}{m}(E - V(q))}$ Recall: our Lagrangian gives reparameterization invariant E-L eqns!

27 April 2005

The Ubiquity of Geodesic Motion

We've seen that many classical systems trace out paths that are geodesics: paths $q:[t_0,t_1] \to Q$ that are critical points of

$$S(q) = \int_{t_0}^{t_1} \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\; dt$$

which is proper time when (Q,g) is a Lorentzian manifold, or arclength when (Q,g) is a Riemannian manifold:

1) $g(q): T_qQ \times T_qQ \longrightarrow \mathbb{R}$

$$(v,w) \longmapsto g(v,w)$$

is bilinear

2) W.r.t. some basis of T_qQ

$$g(v,w) = \delta_{ij} v^i w^j$$

3) $g(q)$ varies smoothly with $q \in Q$.

Lorentzian manifolds represent spacetime, Riemannian manifolds represent space.

We've seen:

1) In the geometric optics approximation, light in $Q = \mathbb{R}^n$ acts like particles tracing out geodesics in the metric

$$g_{ij} = n(q)^2 \delta_{ij}$$

where $n : Q \to (0,\infty)$ is index of refraction.

2) Jacobi saw that a particle in $Q = \mathbb{R}^n$ in some potential $V : Q \to \mathbb{R}$ traces out geodesics in the metric

$$g_{ij} = \frac{2}{m}(E-V)\,\delta_{ij}$$

if the particle has energy E (where $V < E$)

$V > E \iff$ classically forbidden region.

3) A free particle in general relativity traces out a geodesic on a Lorentzian manifold (Q,g).

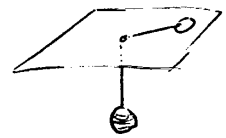
In fact all 3 can be generalized to cover every problem we've discussed!

1') Light on any Riemannian manifold (Q,g) with index of refraction $n : Q \to (0,\infty)$ traces out geodesics in the metric $h = n^2 g$.

2') A particle on a Riemannian manifold (Q,g) with potential $V : Q \to \mathbb{R}$ trace out geodesics w.r.t. the metric

$$h = \frac{2}{m}(E-V)\,g$$

if it has energy E. Lots of physical systems can be described this way:

Atwood machine, , rigid rotating body ($Q = SO(3)$), tops, etc.

— all of these have Lagrangian which is quadratic fn of velocity + some function of position, so they all fit into this framework.

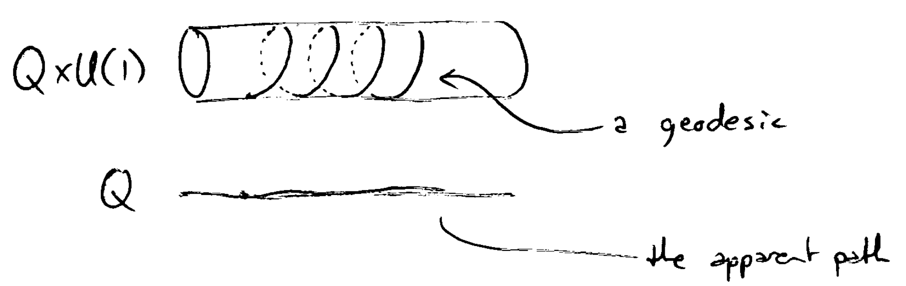
3') <u>Kaluza-Klein theory</u>: a particle with charge e on a Lorentzian manifold (Q,g) in an electromagnetic vector potential follows a path with:

$$\ddot{q}_i = -\Gamma_{ijk}\dot{q}^j\dot{q}^k + \frac{e}{m}F_{ij}\dot{q}^j$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i .$$

This is actually geodesic motion on $Q \times U(1)$ where $U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$ is a circle, & $Q \times U(1)$ is given a cleverly designed metric built from g & A



where the amount of "spiralling" – the velocity in $U(1)$ direction is $\frac{e}{m}$.
The metric h on $Q \times U(1)$ is built from g & A in a very simple way. Let's pick coordinates x^i on Q where $i \in \{0, \ldots, n\}$ since we're in $(n+1)$-dim spacetime, & θ is our local coordinate on S^1. The components of h are

$$h_{ij} = g_{ij} + A_i A_j$$

$$h_{i\theta} = h_{\theta i} = -A_i$$

$$h_{\theta\theta} = 1$$

[illegible] g_{ij}

$h_{i\theta} = h_{\theta i} = +A_i$

Working out the equations for a geodesic in this metric we get:

Christoffel symbols of g

$$\ddot{q}_i = -\Gamma_{ijk}\dot{q}^j\dot{q}^k + \frac{e}{m}F_{ij}\dot{q}^j$$

really a $\dot{q}^\theta$ [illegible] constant

$$\ddot{q}_\theta = 0$$

$$\text{if } \dot{q}_\theta = \frac{e}{m}$$

since F_{ij} is part of the Christoffel symbols for h.

So – every problem we've discussed can be seen as geodesic motion!

29 Apr. 2005

From Lagrangians to Hamiltonians

In the Lagrangian approach we focus on the position & velocity of a particle, & compute what the particle does starting from the Lagrangian $L(q,\dot{q})$, which is a function

$$L : TQ \longrightarrow \mathbb{R}$$

where the tangent bundle is the space of position-velocity pairs. But we're led to consider momentum

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

since the equations of motion say how it changes:

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}$$

In the Hamiltonian approach we focus on position & momentum, & compute what the particle does starting from the energy

$$H = p_i \dot{q}^i - L(q,\dot{q})$$

reinterpreted as a function of position & momentum, the Hamiltonian

$$H : T^*Q \longrightarrow \mathbb{R}$$

where the cotangent bundle is the space of position-momentum pairs. In this approach, position & momentum will satisfy Hamilton's equations:

$$\frac{dq^i}{dt} = \frac{\partial L}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial L}{\partial q^i}$$

where the latter is the Euler Lagrange equation

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}$$

in disguise (minus sign since $H = p_i \dot{q} - L$)

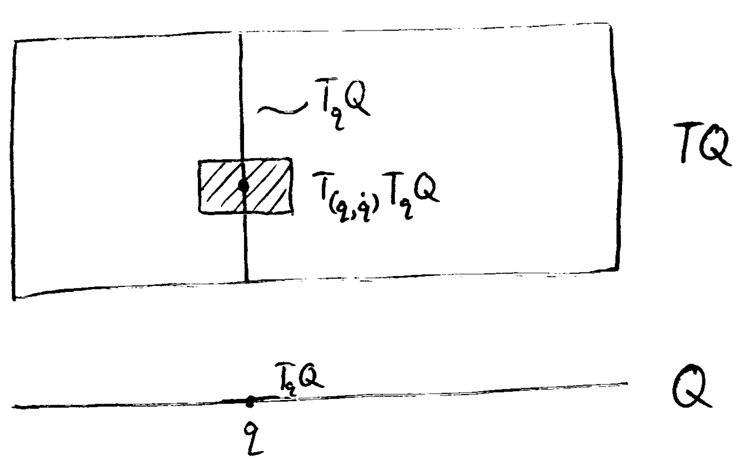
To get to Hamiltonian description, we need to study this map:

$$\lambda : TQ \longrightarrow T^*Q$$
$$(q, \dot{q}) \longmapsto (q, p)$$

where $q \in Q$, $\dot{q}$ is any tangent vector in $T_q Q$ (not the time derivative of something), and p is a cotangent vector in $T_q^* Q := (T_q Q)^*$, given by:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

So: λ is defined using $L : TQ \to \mathbb{R}$. Despite appearances, λ can be defined in a coordinate-free way, as follows.



We want to define "$\frac{\partial L}{\partial \dot{q}^i}$" in a coordinate-free way; it's the "differential of L in the vertical direction" — i.e. the $\dot{q}^i$ directions. We have

$$\pi : TQ \longrightarrow Q$$
$$(q, \dot{q}) \longmapsto q$$

&

$$d\pi : T(TQ) \to TQ$$

has kernel consisting of <u>vertical vectors</u>:

$$VTQ = \ker d\pi \subseteq TTQ.$$

The differential of L at some point $(q,\dot{q}) \in TQ$ is

$$(dL)_{(q,\dot{q})} \in T^*_{(q,\dot{q})} TQ$$

i.e.

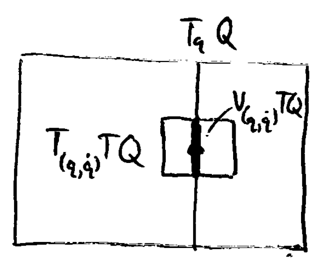
$$dL_{(q,\dot{q})} : T_{(q,\dot{q})} TQ \to \mathbb{R}$$

We can restrict this to $VTQ \subseteq TTQ$, getting

$$f : V_{(q,\dot{q})} TQ \longrightarrow \mathbb{R}$$

But note

$$V_{(q,\dot{q})} TQ = T(T_q Q)$$



& since $T_q Q$ is a vector space,

$$T_{(q,\dot{q})} T_q Q \cong T_q Q$$

in a canonical way.

so f gives a linear map

$$p : T_q Q \longrightarrow \mathbb{R}$$

i.e.

$$p \in T_q^* Q \quad !$$

This is the momentum.