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From the Lagrangian to the Hamiltonian Approach (cont)

Given $L: TQ \rightarrow \mathbb{R}$, we now know a coordinate-free way of describing the map

$$\begin{aligned} \lambda: TQ &\rightarrow T^*Q \\ (q, \dot{q}) &\mapsto (q, p) \end{aligned}$$

given n local coordinates by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

We say L is regular if λ is a diffeomorphism from TQ to some open subset $X \subseteq T^*Q$. In this case we can describe what our system is doing equally well by specifying position & velocity:

$$(q, \dot{q}) \in TQ$$

or position & momentum

$$(q, p) = \lambda(q, \dot{q}) \in X.$$

We call X the phase space of the system. In practice $X = T^*Q$, & then L is said to be strongly regular.

Examples: A particle in a Riemannian manifold (Q, g) in a potential $V: Q \rightarrow \mathbb{R}$ has Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j - V(q)$$

Here

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij} \dot{q}^j$$

so

$$\lambda(q, \dot{q}) = (q, mg(\dot{q}, -))$$

so L is strongly regular since

$$\begin{aligned} T_q Q &\longrightarrow T_q^* Q \\ v &\longmapsto g(v, -) \end{aligned}$$

is 1-1 & onto, i.e. the metric is nondegenerate.

Example: A general relativistic particle with charge e in an electromagnetic vector potential A has Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j - e A_i \dot{q}^i$$

& thus

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij} \dot{q}^j + e A_i$$

This L is still strongly regular, but now each map

$$\begin{aligned} \lambda|_{T_q Q} : T_q Q &\longrightarrow T_q^* Q \\ \dot{q} &\longmapsto mg(\dot{q}, -) + eA(q) \end{aligned}$$

is affine rather than linear.

Example: The free general relativistic particle with reparameterization-invariant Lagrangian:

$$L(q, \dot{q}) = m \sqrt{g_{ij} \dot{q}^i \dot{q}^j}$$

This is terrible from the perspective of regularity properties — not differentiable when $g_{ij} \dot{q}^i \dot{q}^j$ vanishes, & undefined when

this quantity is negative! Where it's defined,

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{mg_{ij}\dot{q}^j}{\|\dot{q}\|}$$

(where \dot{q} is timelike), we can ask about regularity. Alas, the map λ is not 1-1 where defined since multiplying \dot{q} by some number has no effect on p ! (This is related to the reparameterization invariance — this always happens with reparameterization-inv. Lagrangians)

Example: Here's a Lagrangian that's regular but not strongly regular. Let $Q = \mathbb{R}$ &

$$L(q, \dot{q}) = f(\dot{q})$$

so that

$$p = \frac{\partial L}{\partial \dot{q}} = f'(\dot{q})$$

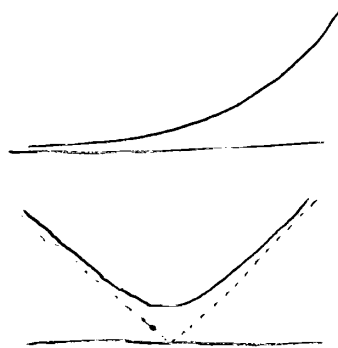
This will be regular but not strongly so if $f': \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism from \mathbb{R} to some proper subset $U \subset \mathbb{R}$. For example, take $f(\dot{q}) = e^{\dot{q}}$ so $f': \mathbb{R} \xrightarrow{\sim} (0, \infty) \subset \mathbb{R}$.

So

$$L(q, \dot{q}) = e^{\dot{q}}$$

or
$$L(q, \dot{q}) = \sqrt{1 + \dot{q}^2}$$

etc.



(slope is positive)

(slope is between -1 & 1)

Now let's assume L is regular, so

$$\lambda: TQ \xrightarrow{\sim} X \subseteq T^*Q$$

$$(q, \dot{q}) \longmapsto (q, p)$$

This lets us have the best of both worlds: we can identify TQ with X using λ . This lets us treat q^i, p^i, L, H , etc. all as functions on X (or TQ), thus writing

$$\dot{q}^i \quad (\text{fn on } TQ)$$

for the function

$$\dot{q}^i \circ \lambda^{-1} \quad (\text{fn on } X)$$

In particular

$$\dot{p}_i := \frac{\partial L}{\partial \dot{q}^i} \quad (\text{Euler-Lagrange eqn.})$$

which is really a fn on TQ , will be treated as a fn on X .

Now, let's calculate:

$$dL = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i$$

$$= \dot{p}_i dq^i + p_i d\dot{q}^i$$

while

$$dH = d(p_i \dot{q}^i - L)$$

$$= \dot{q}^i dp_i + p_i d\dot{q}^i - dL$$

$$= \dot{q}^i dp_i + p_i d\dot{q}^i - (\dot{p}_i dq^i + p_i d\dot{q}^i)$$

$$= \dot{q}^i dp_i + \dot{p}_i dq^i$$

$$dH = \dot{q}^i dp_i - \dot{p}^i dq_i$$

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Assume the Lagrangian $L: TQ \rightarrow \mathbb{R}$ is regular, so

$$\begin{aligned} \lambda: TQ &\xrightarrow{\sim} X \subseteq T^*Q \\ (q, \dot{q}) &\longmapsto (q, p) \end{aligned}$$

is a diffeomorphism. This lets us regard both L and the Hamiltonian $H = p_i \dot{q}^i$ as functions on the phase space X , and use (q^i, \dot{q}^i) as local coordinates on X . As we saw last time, this gives us

$$\begin{aligned} dL &= \dot{p}_i dq^i + p_i d\dot{q}^i \\ dH &= \dot{q}^i dp_i - \dot{p}_i dq^i. \end{aligned}$$

But we can also work out dH directly, this time using local coordinates (q^i, p_i) , to get

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i.$$

Since dp_i, dq^i form a basis of 1-forms, we conclude:

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = - \frac{\partial H}{\partial q^i}}$$

HAMILTON'S
EQUATIONS

Though \dot{q}^i and p_i are just functions of X , when the E-L equations hold for some path $q: [t_0, t_1] \rightarrow Q$, they will be the time derivatives of q^i and p_i . So when the E-L equations hold, Hamilton's equations describe the motion of a point $x(t) = (q(t), p(t)) \in X$. In fact, Hamilton's equations are just the Euler-Lagrange equations in disguise: the equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

really just lets us recover the velocity \dot{q} as a function of q & p , inverting the formula

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

which gave p as a fn. of q & \dot{q} . So we get a formula for

$$\begin{aligned} \lambda^{-1}: X &\longrightarrow TQ \\ (q, p) &\longmapsto (q, \dot{q}) \end{aligned}$$

Given this, the other Hamilton eqn

$$\dot{p}_i = - \frac{\partial H}{\partial q^i}$$

is secretly the E-L eqn

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \text{or} \quad \dot{p}_i = \frac{\partial L}{\partial q^i}$$

These are the same because $\frac{\partial H}{\partial q^i} = \frac{\partial}{\partial q^i} (p_i \dot{q}^i - L) = - \frac{\partial L}{\partial q^i}$.

Example: A particle in $Q = \mathbb{R}^n$ in a potential $V: \mathbb{R}^n \rightarrow \mathbb{R}$.

~ This has Lagrangian $L(q, \dot{q}) = \frac{m}{2} \|\dot{q}\|^2 - V(q)$, which gives

$$p = m\dot{q} \quad \text{so} \quad \dot{q} = \frac{p}{m} \quad \left(\text{really: } \dot{q}^i = \frac{g^{ij} p_j}{m} \right)$$

and Hamiltonian

$$\begin{aligned} H(q, p) &= p_i \dot{q}^i - L = \frac{1}{m} \|p\|^2 - \left(\frac{\|p\|^2}{2m} - V \right) \\ &= \frac{1}{2m} \|p\|^2 + V(q). \end{aligned}$$

So Hamilton's equations say

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \Rightarrow \quad \dot{q} = \frac{p}{m}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \Rightarrow \quad \dot{p} = -\nabla V$$

~ The first just recovers \dot{q} as a function of p ; the second is $F = ma$.

~ Hamilton's eqns push us toward the viewpoint where p & q have equal status as coordinates on the phase space X . Soon, we'll drop the requirement that $X \subseteq T^*Q$ where Q is a configuration space. X will just be a manifold equipped with enough structure to write down Hamilton's eqns starting from any $H: X \rightarrow \mathbb{R}$.

The coordinate-free description of this structure is the major 20th century contribution to mechanics: a symplectic structure.

Hamilton's equations from the Principle of Least Action

Before, we obtained the E-L eqns by associating an "action" S to any $q: [t_0, t_1] \rightarrow Q$ and setting $\delta S = 0$. Now let's get Hamilton's eqns directly by assigning an action S to any path $x: [t_0, t_1] \rightarrow X$ and setting $\delta S = 0$. Note: we don't impose any relation between p & q, \dot{q} ! The relation will follow from $\delta S = 0$.

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Let \mathcal{P} be the space of paths in the phase space X and define the action

$$S: \mathcal{P} \rightarrow \mathbb{R}$$

by

$$S(x) = \int_{t_0}^{t_1} (p_i \dot{q}^i - H) dt$$

where $p_i \dot{q}^i - H = L$. More precisely, write our path x as $x(t) = (q(t), p(t))$ and let

$$S(x) = \int_{t_0}^{t_1} \left(p_i(t) \frac{d}{dt} q^i(t) - H(q(t), p(t)) \right) dt$$

(we write $\frac{d}{dt}q^i$ instead of \dot{q}^i to emphasize that we mean the time derivative rather than a coordinate in phase space.)

Let's show $\delta S = 0 \iff$ Hamilton's equations:

$$\begin{aligned}
 \delta S &= \delta \int (p_i \dot{q}^i - H) dt \\
 &= \int (\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \delta H) dt \\
 &= \int (\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \delta H) dt \quad \left. \begin{array}{l} \text{Int. by parts} \\ \leftarrow \end{array} \right. \\
 &= \int \left(\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt \\
 &= \int \left(\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) + \delta q^i \left(-\dot{p}_i - \frac{\partial H}{\partial q^i} \right) \right) dt
 \end{aligned}$$

This vanishes $\forall \delta x = (\delta q, \delta p)$ if and only if Hamilton's equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad p_i = - \frac{\partial H}{\partial q^i}$$

hold.

We've seen two "principles of least action":

- 1) for paths in configuration space Q , $\delta S = 0 \implies$ E-L eqns
- 2) for paths in phase space X , $\delta S = 0 \implies$ Hamilton's eqns

Additionally, since $X \subseteq T^*Q$, we might consider a third version based on paths in position-velocity space TQ . But when our

Lagrangian is regular, we have a diffeomorphism $\lambda: TQ \xrightarrow{\sim} X$, so this third principle of least action is just a reformulation of (2). However, the really interesting principle of least action involves paths in the extended phase space where we have an additional coordinate for time: $X \times \mathbb{R}$.

Recall the action

$$\begin{aligned} S(x) &= \int (p_i \dot{q}^i - H) dt \\ &= \int p_i \frac{dq^i}{dt} dt - H dt \\ &= \int p_i dq^i - H dt \end{aligned}$$

We can interpret the integrand as a 1-form

$$\beta = p_i dq^i - H dt$$

on $X \times \mathbb{R}$, which has coordinates p_i, q^i, t . So any path

$$x: [t_0, t_1] \longrightarrow X$$

gives a path

$$c: [t_0, t_1] \longrightarrow X \times \mathbb{R}$$

$$t \longmapsto (x(t), t)$$

and the action becomes the integral of a 1-form over a curve:

$$S(x) = \int p_i dq^i - H dt = \int_c \beta$$