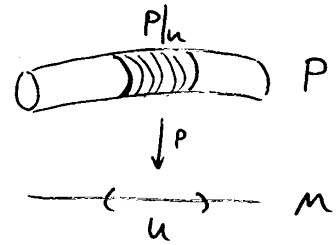
12 April 2005

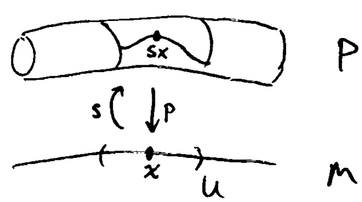
Trivializations & Connections

Suppose $p: P \to M$ is a principal G-bundle. A <u>trivialization</u> <u>over</u> the open set $U \subseteq M$ is an isomorphism of principal G-bundles:

$$\begin{tikzcd} P|_U \arrow[rr, "t", "\sim"'] \arrow[dr, "p"'] & & U \times G \arrow[dl, "p"] \\ & U & \end{tikzcd}$$



(commutes & t preserves action of G). A <u>section</u> over $U \subseteq M$ is a map $s: U \to P|_U$ s.t. $p \circ s = 1_U$:

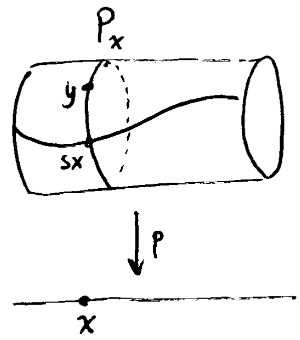


<u>Thm</u> – There's a 1-1 correspondence between trivializations of a principal bundle over U & sections of it over U. (Note: not true for vector bundles!)

<u>Pf</u> – Given a section $s: U \to P|_U$ we define a trivialization $t: P|_U \to U \times G$ as follows:

$$t(y) = (py, \, y/sx) \in U \times G$$

where $y/sx \in G$ is the unique elt such that $(y/sx)sx = y$. (recall that $sx, y \in P_x$ and the fiber P_x is a G torsor)



$py = x$.

Conversely, given a trivialization $t: P|_U \xrightarrow{\sim} U \times G$, we get a section $s: U \longrightarrow P|_U$ by

$$s(x) = t^{-1}(x, 1).$$

These are inverse constructions. ∎

Let's describe a connection on a principal bundle P using a local section. Recall a connection on P is a differential form A on P valued in $\mathfrak{g}$:

$$A \in \Omega^1(P, \mathfrak{g})$$

satisfying:

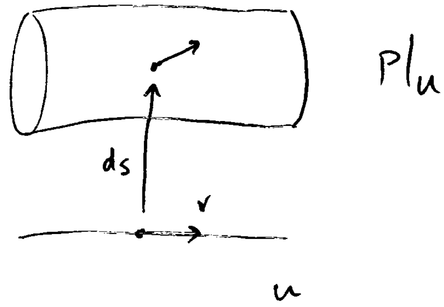
1) A is smooth (implicit in notation "$\Omega^1(M, \mathfrak{g})$")

2) $A|_{V_y} = \alpha : V_y \xrightarrow{\sim} \mathfrak{g}$.

3) $A(dg(v)) = \mathrm{Ad}(g) A(v) \qquad v \in T_y P \quad g \in G$

Given a local section $s: U \longrightarrow P|_U$, we can turn A into a $\mathfrak{g}$-valued 1-form on U:

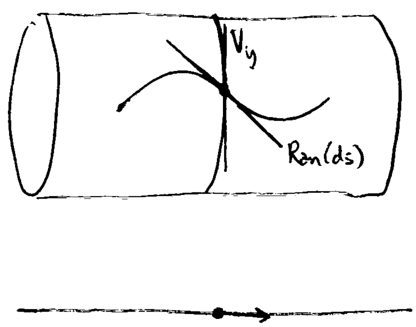
$$\underline{A}(v) = A(ds(v)) \qquad v \in T_x U$$



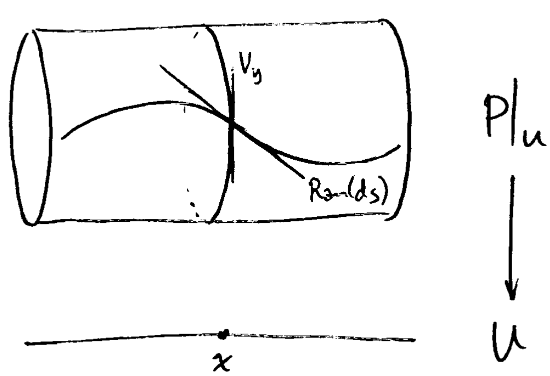
In fact $\underline{A}$ knows all about A:

<u>Thm</u> – Given any $B \in \Omega^1(U, \mathfrak{g})$, there exists a unique connection A on $P|_U$ such that $B = \underline{A}$ (having fixed a section s of P over U to define $\underline{A}$).

<u>Idea</u>:



<u>Proof</u>: If $y = s(x)$ for some $x \in U$, I claim we know what A must do to tangent vectors in $T_y P$



We have $$T_y P = V_y \oplus \mathrm{Ran}(ds) \qquad \text{(check!)}$$

and we know $A(v) = \alpha(v)$ if $v \in V_y$, and $A(ds(w)) = \underline{A}(w)$ if $v = ds(w)$ is in $\mathrm{Ran}(ds)$. What if $y \neq sx$? Why do we still know what A does to T_yP? Answer: condition 3),

$$A(dg(v)) = \mathrm{Ad}(g)\, A(v) \qquad v \in T_yP$$

determines A on $T_{gy}P$ in terms of A on T_yP, so as soon as we know A on $sx \in P$, we know A at all points y in the same fiber. This proves A is unique.

To prove A exists we need to check that this procedure gives a well-defined 1-form on P which satisfies properties 1) – 3). ▨

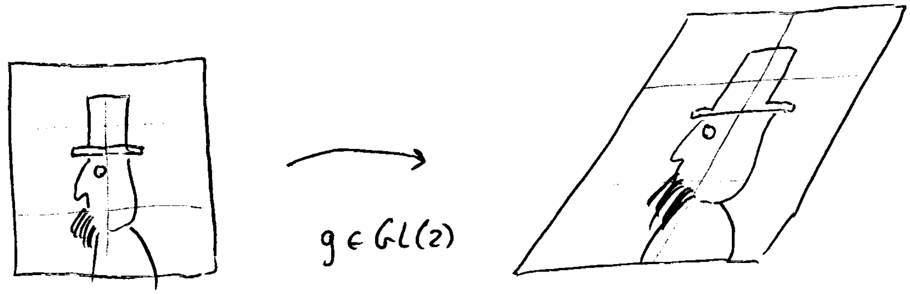
So locally we can pretend a connection on P is a $\mathfrak{g}$-valued 1-form on M – but only over some open set U over which a section is chosen

Associated Bundles

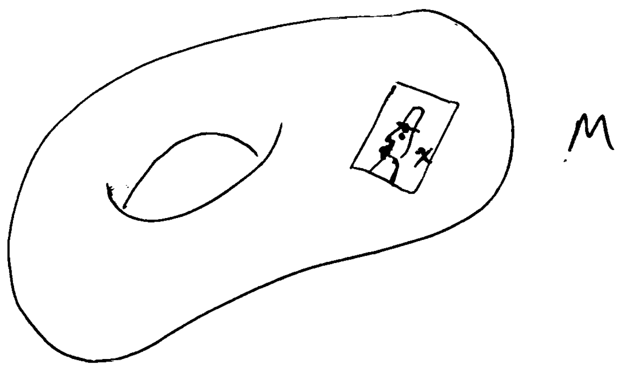
Given a principal G-bundle $p: P \to M$ and given a manifold F on which G acts, we can form a bundle $P \times_G F$ over M whose fibers look like F.

Example: M is a 2-dim manifold, $P = FM$ is the frame bundle of M (whose fiber over $x \in M$ consists of bases for T_xM), $G = GL(2)$, & $F = \{$pictures of Abraham Lincoln in $\mathbb{R}^2\}$

How does $GL(2)$ act on F?



By linear transformations, in the obvious way. From these data we should be able to concoct an "associated bundle" $P \times_G F \to M$, whose points look like:



pictures of Lincoln in tangent spaces of M. Note: given a picture in $\mathbb{R}^2$ & a point in the frame bundle FM (a point $x \in M$ together with a basis for T_xM, i.e. an iso $f: \mathbb{R}^2 \xrightarrow{\sim} T_xM$) we get a picture in a tangent space of M. So, we get a point in $P \times_G F$ from a point in P & a point in F.

In general, we copy this and define

$$P \times_G F = \frac{P \times F}{(gp, gf) \sim (p, f)}$$

since we get the <u>same</u> picture of Lincoln in T_xM from <u>different</u> choices of (a picture of Lincoln in $\mathbb{R}^2$, a frame at $x \in M$) since applying the same element $g \in GL(2)$ to each item gives an unchanged picture in T_xM

$$(pg, f) \sim (p, gf)$$

$$v\alpha \otimes w = v \otimes \alpha w$$

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Associated Bundles, Cont.

Suppose we have:

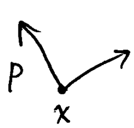
- a Lie group G
- a principal G-bundle $p: P \to M$
- a G-space F, i.e. a manifold on which G acts (smoothly).

From these we form the <u>associated bundle</u>

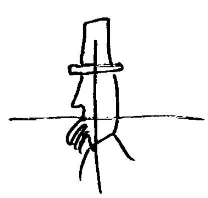
$$E = P \times_G F$$

$$:= \frac{P \times F}{\{(p,f) \sim (gp, gf)\}}$$

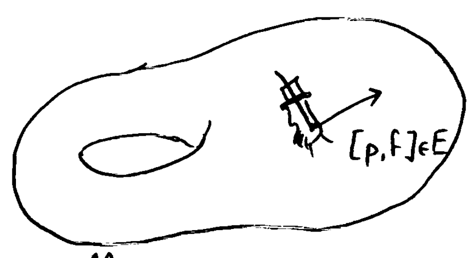
A point in P is like a "frame"; a point in F is like a "figure", & a point in E is like a "figure at a point of M":



$p \in P$



$f \in F$



M

where $[p,f] \in E$ is an equivalence class of pairs $(p,f) \in P \times F$.
Usually people turn the left action of G into a right action by

$$pg := g^{-1}p.$$

& we will do this now, so we get

$$\begin{aligned} E &= P \times_G F \\ &= \frac{P \times F}{(p,f) \sim (pg^{-1}, gf)} \end{aligned}$$

or if $q = pg^{-1}$

$$= \frac{P \times F}{(qg, f) \sim (q, gf)}.$$

<u>Theorem</u>: E is a locally trivializable bundle over M, i.e. for any $x \in M$ there's an open $U \ni x$ s.t. we have

$$\begin{tikzcd} E|_U \arrow[rr, "\sim", "\tau"'] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

$$[q,f] \mapsto p(q)$$

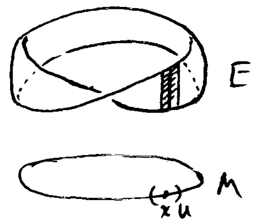
for some diffeomorphism τ, where

$$E|_U := \{[q,f] \in E : p(q) \in U\}$$

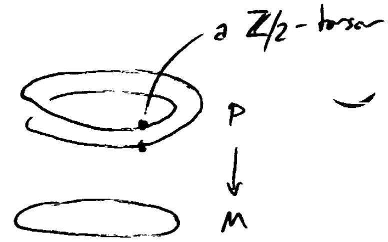
E.g. the Möbius bundle

$G = \mathbb{Z}/2$

$F = (0,1)$



is associated to the principal bundle:



identify two copies of F sitting above x.

$$\frac{\mathbb{Z}/2 \times (0,1)}{\mathbb{Z}/2} \cong \frac{P_x \times F}{\mathbb{Z}/2} = E_x$$

<u>Proof:</u> We get τ using the definition of "principal G-bundle" which says that $p: P \to M$ is locally trivially: for any $x \in M$ $\exists$ nbhd U s.t.

$$\begin{tikzcd}
P|_U \arrow[rr, "\sim", "t"'] \arrow[dr] & & U \times G \arrow[dl] \\
& U &
\end{tikzcd}$$

commutes, where t is a diffeomorphism preserving the right action of G: $t(qg) = t(q)g$. Recall

$$E = P \times_G F$$

so

$$E|_U = P|_U \times_G F$$

So first define

$$\begin{tikzcd}
P|_U \times F \arrow[r, "t \times 1"] \arrow[rr, bend right, "\tau"'] & U \times G \times F \arrow[r, "\alpha"] & U \times F
\end{tikzcd}$$

$$(x, g, f) \longmapsto (x, gf)$$

& check that $\underline{\tau}$ factors through $P|_U \times_G F$ giving our desired $\tau : P|_U \times_G F \longrightarrow U \times F$.

<u>Check:</u>

$$\begin{array}{ccc} \underline{\tau}(qg, f) & \overset{?}{=} & \underline{\tau}(q, gf) \\ \| & & \| \\ \alpha(t(qg), f) & & \alpha(t(q), gf) \\ \| & & \\ \alpha(t(q)g, f) & = & \checkmark \end{array}$$

since $\alpha(zg, f) = \alpha(z, gf)$. ∎

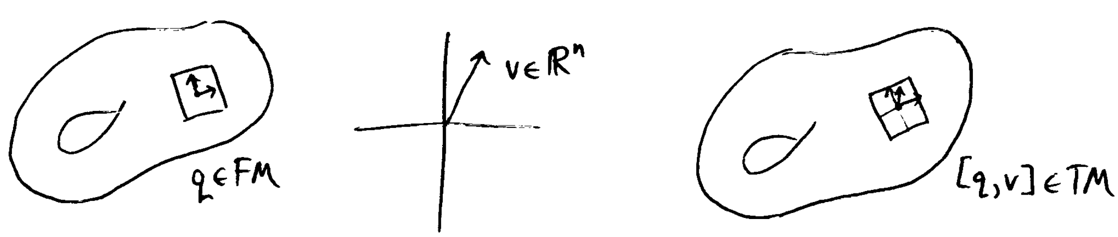
Examples of associated bundles:

1) If F is a vector space and G acts on F as linear transformations (i.e. we have a representation of G on F) then $E = P \times_G F$ is a vector bundle: its fibers

$$E_x = \{ [q, f] : p(q) = x \}$$

will be vector space. In particular:

2) If $P = FM$ is the frame bundle, which is a principal $GL(n)$-bundle, we can take $F = \mathbb{R}^n$ with the obvious representation of $GL(n)$ & get a vector bundle $FM \times_{GL(n)} \mathbb{R}^n$, which is just the tangent bundle TM of M.



3) In the same context we can instead use the dual of the obvious rep., which is a rep on $(\mathbb{R}^n)^*$ & get the <u>cotangent</u> bundle T^*M, which has fiber

$$T_x^*M = (T_xM)^*$$

More generally, $GL(n)$ has a rep on

$$(\mathbb{R}^n)^{\otimes p} \otimes (\mathbb{R}^{n*})^{\otimes q}$$

which gives the vector bundle of (p,q) tensors. Or, use the pth exterior power

$$\Lambda^p(\mathbb{R}^n)^*$$

& get the vector bundle of p-forms, whose sections form the space $\Omega^p(M)$.