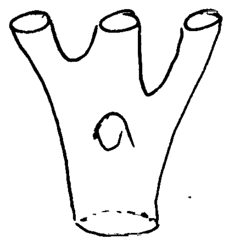
8 Feb 2005

<u>Thm</u> — Suppose G is a finite group. Then we get a 2d TQFT

$$Z : 2\mathrm{Cob} \to \mathrm{Vect}$$

from the semisimple algebra $\mathbb{C}[G]$ (the group algebra of G). Moreover, given a spacetime $M : S \to S'$ in 2Cob:



$$\begin{array}{c} S \\ \downarrow M \\ S' \end{array}$$

we can compute the partition function

$$Z(M) : Z(S) \longrightarrow Z(S')$$

as a path integral (a finite sum) over flat G-connections on M.

<u>Proof</u> — We'll just do the case where M is a closed (oriented) 2-manifold, so that $Z(M) : \underset{\mathbb{C}}{\underset{\cong}{Z(\emptyset)}} \longrightarrow \underset{\mathbb{C}}{\underset{\cong}{Z(\emptyset)}}$

is just a number.

<u>Recall</u>: a (finite-dim, associative) algebra is <u>semisimple</u> if the "Killing form"

$$g(a,b) = \mathrm{tr}(L_a L_b) \qquad \begin{array}{l} L_a : A \to A \\ b \mapsto ab \end{array}$$

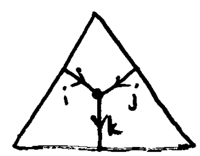
is nondegenerate. To compute $Z(M)$ we pick a basis $e^i \in A$ and define

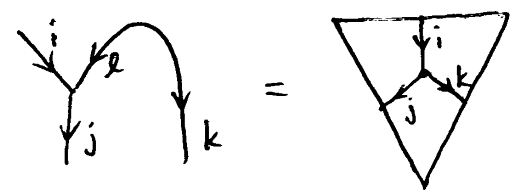
$$e^i e^j = m^{ij}_k e^k \qquad m^{ij}_k \in \mathbb{C}$$

We raise and lower indices on m^{ij}_k using the metric:

$$g^{ij} = g(e^i, e^j)$$

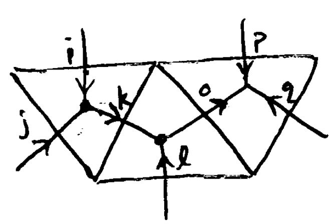
& inverse metric g_{ij} s.t. $g_{ij} g^{jk} = \delta^k_i$. E.g.: we draw

m^{ij}_k 

$m^i_{jk} = m^{il}_j g_{lk}$ 

& similarly for m_{ijk} & m^{ijk}.

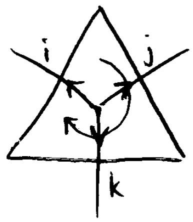
To compute $Z(M)$ we triangulate M, draw a "Feynman Diagram" in the dual 1-skeleton:



$$m^{ji}_k \, m^{lk}_o \, m^{opq}$$

Then: label edges of the Feynman diagrams with indices; write a copy of m for each triangle (using incoming arrows as superscripts, outgoing as subscripts and going around ~~clockwise~~ using the orientation on M.

e.g. :



$$m_{ijk} = m_{jki} = m_{kij}$$

and use Einstein summation convention to get a number $Z(M)$.

Now, when $A = \mathbb{C}[G]$, let's show the sum over indices gives a sum over flat G-connections on (the <u>triangulated</u>) M. Let's use the basis $e^i \in \mathbb{C}[G]$ corresponding to the group elements $i \in G$. Then

$$\underbrace{e^i e^j}_{\text{mult. in } \mathbb{C}[G]} = e^{\overbrace{ij}^{\text{mult. in } G}}$$

so

$$e^i e^j = m^{ij}_k e^k$$

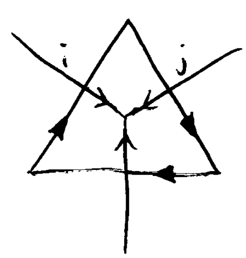
where

$$m^{ij}_k = \delta^{ij}_k = \begin{cases} 1 & ij = k \\ 0 & ij \neq k \end{cases}$$

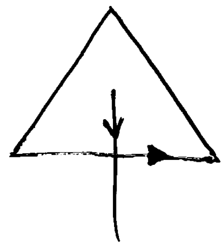
Toby: "on the left there's an implicit <u>comma</u>; on the right an implicit multiplication between i & j."

(cute notation)

When we sum over indices to compute $Z(M)$, we're summing over <u>connections</u> on the triangulated M:



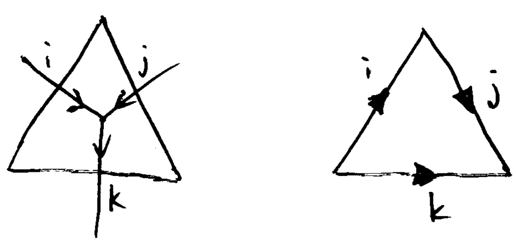
where we use the arrows on our Feynman diagram to orient the edges of M:



(via this rule)

& interpret the group elements $(i, j, k, \ldots)$ as a connection on the directed graph made of triangle edges. We're summing over all connections.

Since $m^{ij}_k = 0$ unless $ij = k$, we get a zero contribution to $Z(M)$ unless the connection is flat on this triangle



What about the other cases: m^{ijk}, m^i_{jk}, & m_{ijk}? We'll see these all vanish unless the connection is flat – but let's see exactly what they are. First, let's calculate g^{ij} & g_{ij}:

$$\begin{aligned} g^{ij} = g(e^i, e^j) &= tr(L_i L_j) \\ &= tr(L_{ij}) \\ &= \sum_{k \in G} \langle e^k, L_{ij} e^k \rangle \end{aligned}$$

inner prod s.t. e^i are o.n. basis

$$= \sum_{k \in G} \langle e^k, e^{ijk} \rangle$$

$$= \sum_{k \in G} \delta^{ijk}_{k}$$

$$= \sum_{k \in G} \delta^{ij}_{1} \qquad \begin{array}{c} ijk = k \\ \Updownarrow \\ ij = 1 \end{array}$$

$$= |G| \, \delta^{ij}_{1}$$

$$= |G| \, \delta^{i}_{j^{-1}}$$

The inverse of this matrix is

$$g_{ij} = \frac{1}{|G|} \delta^{1}_{ij} = \frac{1}{|G|} \delta^{j^{-1}}_{i}$$

Using these we get:

$$m^{i}_{jk} = m^{il}_{j} g_{lk} = m^{il}_{j} \cdot \frac{1}{|G|} \delta^{k^{-1}}_{l} = \frac{1}{|G|} m^{ik^{-1}}_{j} = \frac{1}{|G|} \delta^{ik^{-1}}_{j} = \frac{1}{|G|} \delta^{i}_{jk}$$

$$\left(\begin{array}{c} ik^{-1} = j \text{ or get zero} \\ \Updownarrow \\ i = jk \end{array} \right.$$

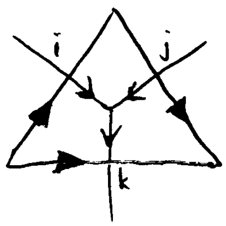
which resembles $m^{ij}_{k} = \delta^{ij}_{k}$ except for the $\frac{1}{|G|}$ — which comes from lowering an index. Similarly:

$$m_{ijk} = \frac{1}{|G|^2} \delta^{1}_{ikj} \qquad \text{(note order of indices!)}$$

$$\& \quad m^{ijk} = |G| \, \delta^{ijk}_{1} \, .$$

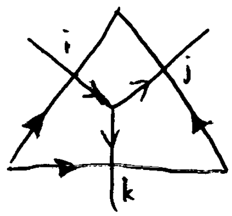
Key point for finishing the proof: In every case we get zero unless the connection is flat:

1



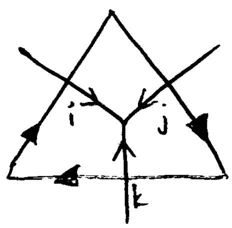
$$m^{ij}_k = 0 \quad \text{unless} \quad ij = k$$
$$m^{ij}_k = \delta^{ij}_k$$

$\frac{1}{|G|}$



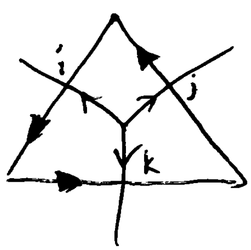
$$m^{i}_{kj} = 0 \quad \text{unless} \quad i = kj$$
$$m^{i}_{kj} = \frac{1}{|G|}\,\delta^{i}_{kj}$$

$|G|$



$$m^{ijk} = 0 \quad \text{unless} \quad ijk = 1$$
$$m^{ijk} = |G|\,\delta^{ijk}_1$$

$\frac{1}{|G|^2}$



$$m_{ijk} = 0 \quad \text{unless} \quad ikj = 1$$
$$m_{ijk} = \frac{1}{|G|^2}\,\delta^{1}_{ikj}$$

∎

In fact: we've seen $Z(M)$ is a sum over flat connections of

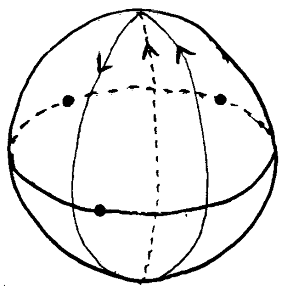
$|G|^{\#}$  $- \#$  $- 2\#$ 

So $Z(M)$ is #(flat G-connections) times this number.

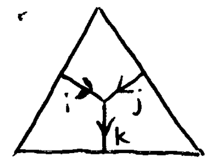
10 February 2005

Examples:

1) $M = S^2$ triangulated with two triangles

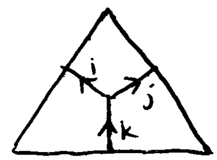


top :



m^{ij}_{k}

bottom

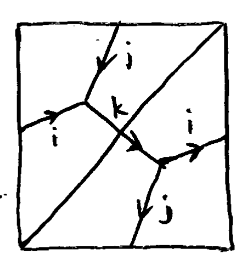


m^{k}_{ij}

So

$$Z(M) = m^{ij}_{k}\, m^{k}_{ij}$$
$$= \frac{1}{|G|}\, \delta^{ij}_{k}\, \delta^{k}_{ij}$$
$$= \frac{1}{|G|}\, |G|^2 = |G|.$$

2) $M = T^2$



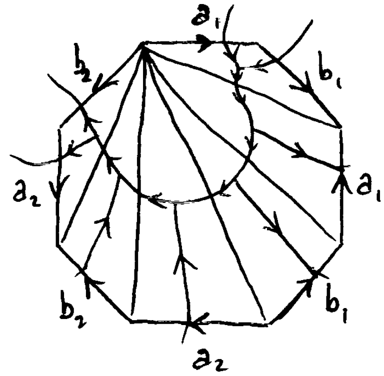
flatness here $\Rightarrow ij = k$

flatness here $\Rightarrow ji = k$

$$Z(M) = m^{ij}_{k}\, m^{k}_{ji}$$
$$= \frac{1}{|G|}\, \delta^{ij}_{k}\, \delta^{k}_{ji}$$
$$= \frac{1}{|G|}\, \left|\{(i,j) \in G^2 : ij = ji\}\right|$$

3) $M = M_g$, the surface of genus g ("g-holed torus").

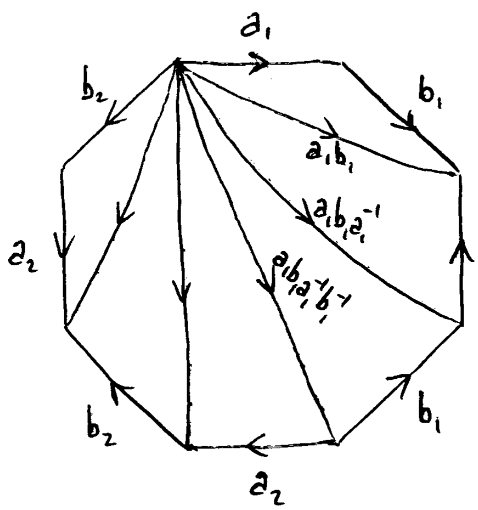
E.g. $g = 2$:



(Note: Suspiciously similar to the Euler characteristic $\chi(M_g) = |V| - |E| + |F| = 2 - 2g$)

We know $Z(M) = \underbrace{|\mathcal{A}_0(M)|}_{\text{flat } G\text{-conns.}} \cdot |G|^{\# \triangle - \# \triangle - 2\# \triangle}$

& hence a flat connection is



$a_1, a_2, b_1, b_2 \in G$ such that

$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1$

(since loop around the octagon is contractible)

So $\mathcal{A}_0(M_g) = \{(a_1, b_1, \ldots, a_n, b_n) : a_1 b_1 a_1^{-1} b_1^{-1} \cdots = 1\} \subseteq G^{2n}$

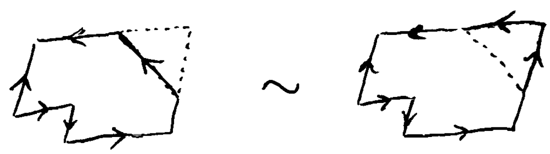
Also $\# Y - \# Y - 2\# Y = -(2g-1)$

So:

$$Z(M_g) = |A_0(M_g)|\,|G|^{-2g+1}$$

where $|A_0(M_g)$ is hard to calculate beyond what we've done. Note this is even true for S^2!

<u>Recall</u>: if we pick a vertex $* \in M$, $\pi_1(M)$ is the group of loops of edges based at $*$, mod simplicial homotopy:



$\pi_1(M_g)$ is generated by $\bar{a}_i, \bar{b}_i$ with only nontrivial relation being $\bar{a}_1\bar{b}_1\bar{a}_1^{-1}\bar{b}_1^{-1} = 1$. So, a guy in $A_0(M_g)$ is just a homomorphism

$$A : \pi_1(M_g) \longrightarrow G$$

given by

$$A(\bar{a}_i) = a_i \quad \& \quad A(\bar{b}_i) = b_i$$

(It's <u>flatness</u> that makes $A(\gamma)$ invariant under homotopies of the loop γ, hence well-defined.)
In short, in this example

$$A_0(M) \cong \hom(\pi_1(M), G)$$

but this relies on the fact that our triangulation just had one vertex! If we switch to a triangulation with more vertices, $\mathcal{A}_0(M)$ gets bigger but $\pi_1(M)$ is unchanged, so $\hom(\pi_1(M), G)$ is unchanged.

But luckily:

<u>Thm</u> — If M is any connected triangulated manifold with chosen vertex $*$, we have

$$\mathcal{A}_0(M)/\mathcal{G}_0(M) \cong \hom(\pi_1(M), G)$$

where $\mathcal{G}_0(M) \leq \mathcal{G}(M)$ is the group of gauge transformations $g : V \to G$ such that $g(*) = 1 \in G$. (We've seen that $\mathcal{G}$ acts on $\mathcal{A}$, but this action preserves $\mathcal{A}_0 \leq \mathcal{A}$. So, the subgroup $\mathcal{G}_0(M) \leq \mathcal{G}(M)$ acts on $\mathcal{A}_0(M)$. $\mathcal{A}_0(M)/\mathcal{G}_0(M)$ is the set of orbits.)

<u>Remark:</u>

$$\mathcal{A}(M) = \{A : E \to G\} \quad \& \quad \mathcal{G}(M) = \{g : V \to G\}$$

but also:

- a connection A is a functor $A : PM \to G$ from the groupoid of paths in M to G
- a gauge transformation g with $gA = A'$ is a natural transformation

$$\begin{tikzcd} PM \arrow[r, bend left, "A"{name=U}] \arrow[r, bend right, "A'"'{name=D}] & G \arrow[Rightarrow, from=U, to=D, "g"] \end{tikzcd}$$

So we'll use the relation between the groupoid PM and the group $\pi_1 M$