

## Equivalence of Categories

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1. Prove that a functor  $F: C \rightarrow D$  is an equivalence iff it is essentially surjective, full and faithful.

*Hint: let me remind you of all the necessary definitions.*

**Definition 1.** A category  $C$  consists of:

- a collection  $\text{Ob}(C)$  of **objects**.
- for any pair of objects  $x, y$ , a set  $\text{hom}(x, y)$  of **morphisms** from  $x$  to  $y$ . (If  $f \in \text{hom}(x, y)$  we write  $f: x \rightarrow y$ .)

equipped with:

- for any object  $x$ , an **identity morphism**  $1_x: x \rightarrow x$ .
- for any pair of morphisms  $f: x \rightarrow y$  and  $g: y \rightarrow z$ , a morphism  $fg: x \rightarrow z$  called the **composite** of  $f$  and  $g$ .

such that:

- for any morphism  $f: x \rightarrow y$ , the **left and right unit laws** hold:  $1_x f = f = f 1_y$ .
- for any triple of morphisms  $f: w \rightarrow x$ ,  $g: x \rightarrow y$ ,  $h: y \rightarrow z$ , the **associative law** holds:  $(fg)h = f(gh)$ .

We usually write  $x \in C$  as an abbreviation for  $x \in \text{Ob}(C)$ . An **isomorphism** is a morphism  $f: x \rightarrow y$  with an **inverse**, i.e. a morphism  $g: y \rightarrow x$  such that  $fg = 1_x$  and  $gf = 1_y$ .

**Definition 2.** Given categories  $C, D$ , a **functor**  $F: C \rightarrow D$  consists of:

- a function  $F: \text{Ob}(C) \rightarrow \text{Ob}(D)$ .
- for any pair of objects  $x, y \in \text{Ob}(C)$ , a function  $F: \text{hom}(x, y) \rightarrow \text{hom}(F(x), F(y))$ .

such that:

- **$F$  preserves identities:** for any object  $x \in C$ ,  $F(1_x) = 1_{F(x)}$ .
- **$F$  preserves composition:** for any pair of morphisms  $f: x \rightarrow y$ ,  $g: y \rightarrow z$  in  $C$ ,  $F(fg) = F(f)F(g)$ .

It's not hard to define identity functors and composition of functors, and to check the left and right unit law and associative law for these.

**Definition 3.** Given functors  $F, G: C \rightarrow D$ , a **natural transformation**  $\alpha: F \Rightarrow G$  consists of:

- a function  $\alpha$  mapping each object  $x \in C$  to a morphism  $\alpha_x: F(x) \rightarrow G(x)$

such that:

- for any morphism  $f: x \rightarrow y$  in  $C$ , this diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \alpha_x \downarrow & & \downarrow \alpha_y \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

With a little thought you can figure out how to compose natural transformations  $\alpha: F \rightarrow G$  and  $\beta: G \Rightarrow H$  and get a natural transformation  $\alpha\beta: F \Rightarrow H$ . We can also define identity natural transformations. Again, it's not hard to check the left and right unit law and associativity for these.

**Definition 4.** Given functors  $F, G: C \rightarrow D$ , a **natural isomorphism**  $\alpha: F \Rightarrow G$  is a natural transformation that has an **inverse**, i.e. a natural transformation  $\beta: G \Rightarrow F$  such that  $\alpha\beta = 1_F$  and  $\beta\alpha = 1_G$ .

It's not hard to see that a natural transformation  $\alpha: F \Rightarrow G$  is a natural isomorphism iff for every object  $x \in C$ , the morphism  $\alpha_x$  is invertible.

**Definition 5.** A functor  $F: C \rightarrow D$  is an **equivalence** if it has a **weak inverse**, that is, a functor  $G: C \rightarrow D$  such that there exist natural isomorphisms  $\alpha: FG \Rightarrow 1_C$ ,  $\beta: GF \Rightarrow 1_D$ .

**Definition 6.** A functor  $F: C \rightarrow D$  is **essentially surjective** if for every object  $x \in D$  there is an object  $\tilde{x} \in C$  such that  $F(\tilde{x}) \cong x$ .

**Definition 7.** A functor  $F: C \rightarrow D$  is **full** if for every pair of objects  $x, y \in C$ , the function  $F: \text{hom}(x, y) \rightarrow \text{hom}(F(x), F(y))$  is onto.

**Definition 8.** A functor  $F: C \rightarrow D$  is **faithful** if for every pair of objects  $x, y \in C$ , the function  $F: \text{hom}(x, y) \rightarrow \text{hom}(F(x), F(y))$  is one-to-one.

*I will be glad to give you further hints if you need them. The fun part is constructing the weak inverse of a functor  $F$  using the fact that it's essentially surjective, full and faithful. This is a categorified version of constructing the inverse of a function using the fact that it's surjective and injective.*