Math 10A
Derek Wise

Limits - Supplemental Homework

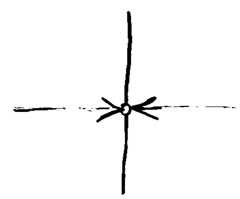
This exercise is designed to show you that we must be careful when calculating limits of functions of more than one variable. Consider the function

$$f(x,y) = \frac{2x^2 y}{x^4 + y^2}$$

defined on all of $\mathbb{R}^2$ except the point $(0,0)$ (since $x^4 + y^2 = 0$ there).

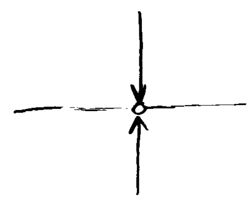
(a) Calculate the limit as we approach along the x-axis. That is, calculate

$$\lim_{x \to 0} f(x,0).$$



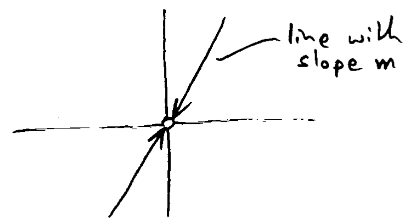
(b) Calculate the limit as we approach along the y-axis:

$$\lim_{y \to 0} f(0,y)$$



(c) More generally, we could approach along any straight line $y = mx$ through the origin. That is, calculate

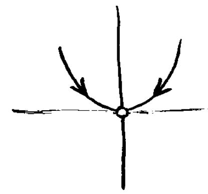
$$\lim_{x \to 0} f(x, mx)$$



In each of parts a, b, & c you should have found that the limit is 0. So we get the same limiting value, 0, as we approach $(0,0)$ along <u>any</u> <u>straight line</u>. From this we might naively conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. <u>But</u>...

(d) Calculate the limit as we approach along the parabola $y = x^2$:

$$\lim_{x \to 0} f(x, x^2)$$



Conclude from this that f has no limit at $(0,0)$.

Here is the key point of this exercise: We can never prove a function has a limit at (x_0, y_0) simply by checking the limiting values for various paths to (x_0, y_0), since no matter how many different approaches we check, there could always still be one that gives a different limiting value. This is why we need the ε-δ definition of limits. On the other hand, to prove a function does <u>not</u> have a limit at (x_0, y_0) it suffices to find two different approaches with different limiting values.