MATH 10A
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Quiz 3

(Key)

1) Find the equation of the tangent plane to the graph of $f(x,y) = xy$ at the point $(1, 2, f(1,2))$. Simplify your answer to the form $z = Ax + By + C$.

$$\frac{\partial f}{\partial x} = y \quad , \quad \frac{\partial f}{\partial y} = x \qquad \text{so} \qquad \frac{\partial f}{\partial x}(1,2) = 2 \quad , \quad \frac{\partial f}{\partial y}(1,2) = 1$$

So the tangent plane is

$$z = f(1,2) + \left[\frac{\partial f}{\partial x}(1,2)\right](x-1) + \left[\frac{\partial f}{\partial y}(1,2)\right](y-2)$$

$$z = 2 + 2(x-1) + 1(y-2)$$

$$\boxed{z = 2x + y - 2}$$

2) Compute the derivative $Df(x,y,z)$ for the function $f: \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x,y,z) = (2xz^2, e^{5yz})$.

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2z^2 & 0 & 4xz \\ 0 & 5ze^{5yz} & 5ye^{5yz} \end{bmatrix}$$

3) Let $\vec{c}(t) = (2\sin \pi t,\ 3\cos \pi t)$ be a path.

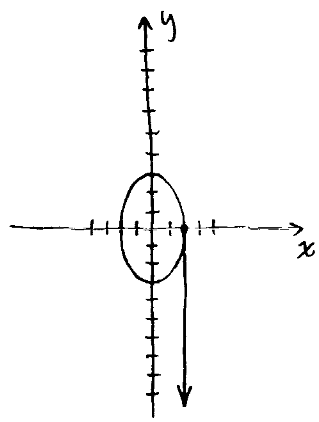
(a) Calculate the velocity vector at time $t = \frac{1}{2}$.

$$\vec{c}\,'(t) = (2\pi \cos \pi t, -3\pi \sin \pi t)$$

$$\vec{c}\,'(\tfrac{1}{2}) = (2\pi \cos \tfrac{\pi}{2}, -3\pi \sin \tfrac{\pi}{2})$$

$$= (0, -3\pi)$$

(b) Sketch the image curve of the path and on your sketch draw in the velocity vector at time $t = \frac{1}{2}$, which you calculated in part (a).



$\vec{c}(\tfrac{1}{2}) = (2, 0)$

$\vec{c}\,'(\tfrac{1}{2}) = -3\pi \hat{\jmath}$

4) (a) Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ and $g:\mathbb{R}^3 \to \mathbb{R}^2$ be differentiable functions. Assume that $g(1,0,5) = (2,7)$ and that

$$Dg(1,0,5) = \begin{bmatrix} 2 & 5 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad Df(2,7) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find $D(f\circ g)(1,0,5)$.

$$\begin{aligned} D(f\circ g)(1,0,5) &= Df(g(1,0,5))\,Dg(1,0,5) \\ &= Df(2,7)\,Dg(1,0,5) \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{aligned}$$

(b) Let $h(x,y) = f(x, u(x,y), v(y))$. Find $\frac{\partial h}{\partial x}$ using the chain rule.

Write $g(x,y) = (p(x), u(x,y), v(y))$ where $p(x) = x$.

Then $h = f\circ g$ and we have

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial p}\underbrace{\frac{dp}{dx}}_{=1} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=0} = \boxed{\frac{\partial f}{\partial p} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}}$$

(Note: we can verify this using the matrix chain rule:

$$\underbrace{\begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}}_{Dh} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}}_{Df} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ 0 & \frac{dv}{dy} \end{bmatrix}}_{Dg} \quad)$$