MATH 10a
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(KEY)

Quiz 4

1) Calculate the following partial derivatives:

(a) $\frac{\partial^2}{\partial y \partial x} x \cos(x+y)$

$$= \frac{\partial}{\partial y}\left(\cos(x+y) - x\sin(x+y)\right)$$

$$= -\sin(x+y) - x\cos(x+y)$$

(b) $\frac{\partial^2 f}{\partial x^2}$ where $f(x,y,z) = e^{2x+3y}$

$$\frac{\partial f}{\partial x} = 2e^{2x+3y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2e^{2x+3y}\right) = 4e^{2x+3y}$$

(c) $\frac{\partial^3}{\partial x \partial y \partial z} xy^2z$

$$= \frac{\partial^2}{\partial x \partial y}(xy^2) = \frac{\partial}{\partial x}(2xy) = 2y$$

2) Find the directional derivative of $f(x,y) = xy^3 + x^2$ at the point $(1,2)$ in the direction $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$.

$$\vec{\nabla} f = (y^3 + 2x,\ 3xy^2)$$

$$\vec{\nabla} f(1,2) = (8+2,\ 12) = (10,12)$$

So $D_{\vec{v}} f(1,2) = \vec{v}\cdot\vec{\nabla} f(1,2) = \left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)\cdot\left(10\hat{\imath} + 12\hat{\jmath}\right)$

$$= \frac{10}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \boxed{\frac{22}{\sqrt{2}}}$$

3) Find the second-order Taylor approximation for the function

$$f(x,y) = e^{-x^2-y^2}$$

at the point $(0,0)$.

$$\frac{\partial f}{\partial x} = -2xe^{-x^2-y^2} \qquad \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = -2ye^{-x^2-y^2} \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-2xe^{-x^2-y^2}\right) \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = -2$$

$$= -2e^{-x^2-y^2} + 4x^2e^{-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -2e^{-x^2-y^2} + 4y^2e^{-x^2-y^2} \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = -2$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(-2xe^{-x^2-y^2}\right)$$

$$= 4xye^{-x^2-y^2} \qquad \frac{\partial^2 f}{\partial y\,\partial x}(0,0) = 0$$

So @ $(0,0)$ the second order Taylor polynomial is

$$p_2(x,y) = f(0,0) + \left[\frac{\partial f}{\partial x}(0,0)\right]x + \left[\frac{\partial f}{\partial y}(0,0)\right]y + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(0,0)\right]x^2 + \left[\frac{\partial^2 f}{\partial y\,\partial x}(0,0)\right]xy + \frac{1}{2}\left[\frac{\partial^2 f}{\partial y^2}(0,0)\right]y^2$$

$$= 1 + 0x + 0y + \tfrac{1}{2}(-2)x^2 + 0xy + \tfrac{1}{2}(-2)y^2$$

$$= \boxed{1 - x^2 - y^2}$$

4) Verify that the function $f(x,y) = \sin(xy)$ has a critical point at $(0,0)$. Use the second-derivative test to classify this critical point as a minimum, maximum, or saddle point.

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y \cos xy \,,\, x \cos xy)$$

So $\vec{\nabla} f(0,0) = (0 \cdot 1, 0 \cdot 1) = \vec{0}$, so $(0,0)$ is critical.

$$\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy \qquad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(y \cos xy)$$

$$= \cos xy - xy \sin xy$$

So: $$D = \left(\frac{\partial^2 f}{\partial x^2}(0,0)\right)\left(\frac{\partial^2 f}{\partial y^2}(0,0)\right) - \left(\frac{\partial^2 f}{\partial y \partial x}(0,0)\right)^2$$

$$= (0)(0) - (1)^2$$

$$= -1$$

$D < 0 \implies (0,0)$ is a saddle point of f.

Bonus: Calculate the acceleration $\vec{a}$ of a particle following the path $\vec{c}(t) = (\sin t, 2\cos 2t, t)$ at time $t = 0$.

velocity $= \vec{c}\,'(t) = (\cos t, -4 \sin 2t, 1)$

$$\vec{c}\,''(t) = (-\sin t, -8\cos 2t, 0)$$

So $\vec{a} = \vec{c}\,''(0) = \boxed{(0, -8, 0)}$