MATH 10a
Derek Wise

Quiz 5

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(KEY)

1) Find the arclength of the path

$$\vec{c}(t) = (\sin 3t, 4t, \cos 3t)$$

for $-\pi \leq t \leq \pi$.

$$\vec{c}\,'(t) = (3\cos 3t, 4, -3\sin 3t)$$

So

$$L = \int_{-\pi}^{\pi} \sqrt{9\cos^2 3t + 16 + 9\sin^2 3t}\; dt$$

$$= \int_{-\pi}^{\pi} \sqrt{9(\cos^2 3t + \sin^2 3t) + 16}\; dt$$

$$= \int_{-\pi}^{\pi} \sqrt{9+16}\; dt$$

$$= 5\int_{-\pi}^{\pi} dt$$

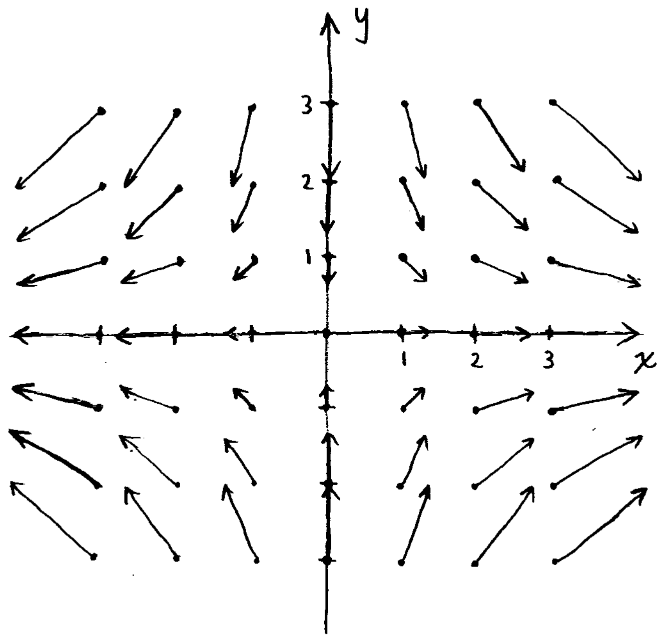
$$= 5(2\pi)$$

$$= \boxed{10\pi}$$

2) Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by

$$\vec{F}(x,y) = x\,\hat{\imath} - y\,\hat{\jmath}$$

(a) Sketch $\frac{1}{3}\vec{F}$.



(Note: I wouldn't expect you to spend as much time on this as I did -- but your picture should be reasonably accurate!)

(b) Show that the curve $\vec{c}(t)$ is a flow line for $\vec{F}$, where

$$\vec{c}(t) = (x_0 e^{t}, y_0 e^{-t}).$$

$$\vec{c}\,'(t) = (x_0 e^{t}, -y_0 e^{-t}) = x_0 e^{t}\,\hat{\imath} - y_0 e^{-t}\,\hat{\jmath}$$

and $$\vec{F}(\vec{c}(t)) = \vec{F}(x_0 e^{t}, y_0 e^{-t}) = x_0 e^{t}\,\hat{\imath} - y_0 e^{-t}\,\hat{\jmath}$$

So $$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$

3) Prove that for any C^2 function $f: \mathbb{R}^3 \to \mathbb{R}$ we have

$$\vec{\nabla} \times (\vec{\nabla} f) = 0.$$

Why do you need f to be of class C^2?

$$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{\imath} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \hat{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \hat{k}$$

and each term is zero, since the mixed partial derivatives of a C^2 function are equal (this is why we need f C^2). Thus:

$$\vec{\nabla} \times (\vec{\nabla} f) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$= \vec{0}.$$

4) Show that the function

$$f(x,y,z,t) = (x-ct)^2$$

satisfies the wave equation:

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x-ct)^2$$

$$= \frac{\partial}{\partial x} 2(x-ct) + 0 + 0$$

$$= 2$$

and

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t} 2(x-ct)(-c)$$

$$= \frac{1}{c^2}(-2c)(-c)$$

$$= 2\frac{c^2}{c^2}$$

$$= 2$$

so $\nabla^2 f = 2 = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$