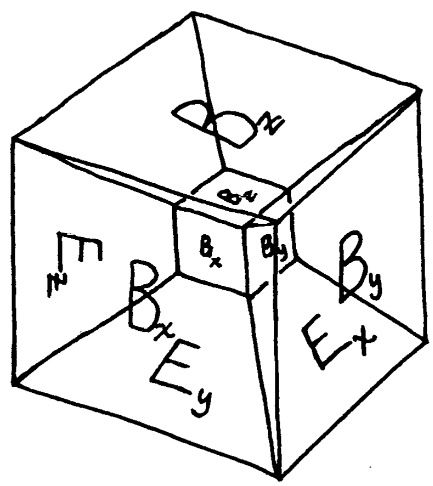
Electricity, Magnetism & Hypercubes



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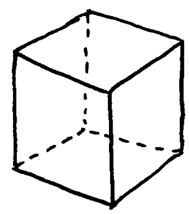
"Atoms of Spacetime"

The simplest way to chop space up into discrete chunks is to chop it into <u>cubes</u>:



But Einstein taught us that we shouldn't think of space & time separately — we should think of time as "the fourth dimension" in a 4d "<u>spacetime</u>"!

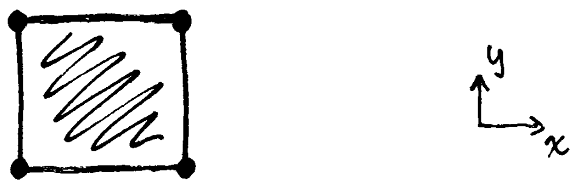
We can chop spacetime up not into cubes but into "hypercubes" — so we need to understand these! ...

How to build a hypercube:

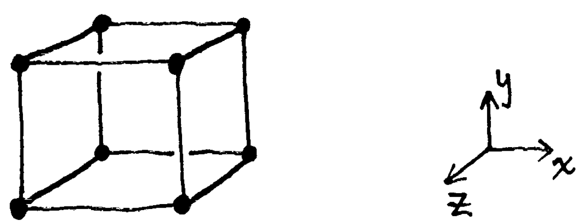
Start with a "0-cube" – a point:

Drag it in the x-direction to get a "1-cube" – a segment:

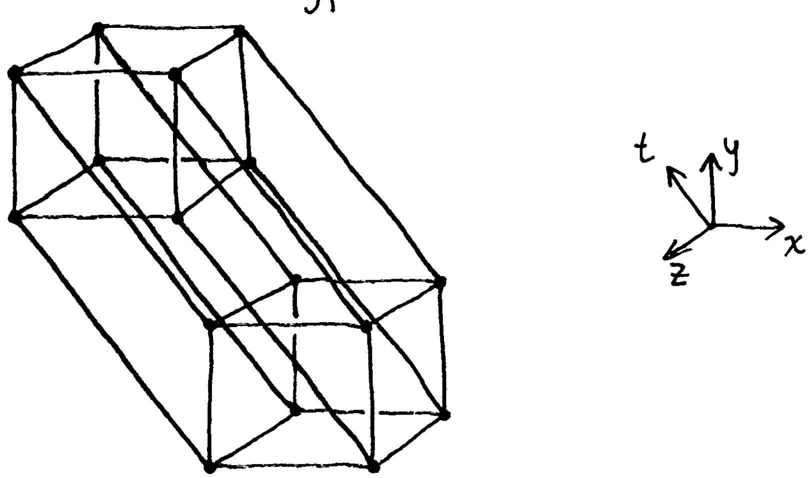
Drag this 1-cube in the y-direction to get a "2-cube" – a square:



Drag this in the z-direction to get a "3-cube" – a cube:



Finally, drag this 3-cube in the t-direction (i.e. wait a while!) to get a "4-cube" – a hypercube:

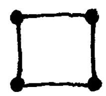


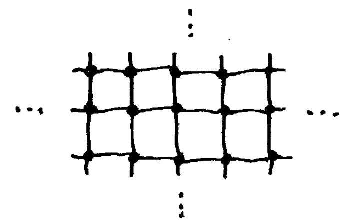
n-Cube Lattices

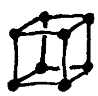
0-cube (point)  1 vertex 0-cube lattice

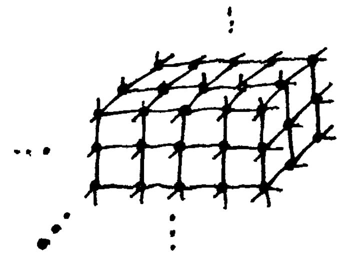
1-cube (segment) 
$2 \times 1 = 2$ vertices
1 edge

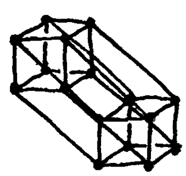
1-cube lattice


2-cube (square) 
$2 \times 2 = 4$ vertices
$2 \times 1 + 2 = 4$ edges
1 2-face


2-cube lattice

3-cube (cube) 
$2 \times 4 = 8$ vertices
$2 \times 4 + 4 = 12$ edges
$2 \times 1 + 4 = 6$ 2-faces
1 3-face


3-cube lattice

4-cube (hypercube) 
$2 \times 8 = 16$ vertices
$2 \times 12 + 8 = 32$ edges
$2 \times 6 + 12 = 24$ faces
$2 \times 1 + 6 = 8$ 3-faces
1 4-face

4-cube lattice (fills 4-dimensional space)

⋮ ⋮ ⋮

In general, if

N^n_m = # of m-faces in an n-cube

we have

$$N^n_m = 2N^{n-1}_m + N^n_{m-1}.$$

This gives something like a modified Pascal's triangle.

1
2 1
4 4 1
8 12 6 1
16 32 24 8 1
32 80 80 40 10 1
64 192 240 160 60 12 1
⋰ ⋮ ⋱

Maxwell's Equations

The fundamental equations which describe all of the phenomena of classical electricity & magnetism are <u>Maxwell's equations</u>. These come naturally in two pairs:

$$\begin{aligned} \nabla\cdot\underbrace{\vec{B}}_{\text{magnetic field}} &= 0 \\ \nabla\times\underbrace{\vec{E}}_{\text{electric field}} + \frac{\partial\vec{B}}{\partial t} &= 0 \end{aligned} \quad \Bigg\} \quad \begin{array}{l}\text{source-independent}\\ \text{or "tautologous" pair}\end{array}$$

$$\begin{aligned} \nabla\cdot\vec{E} &= \underbrace{\rho}_{\text{charge distribution}} \\ \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= \underbrace{\vec{J}}_{\text{current}} \end{aligned} \quad \Bigg\} \quad \begin{array}{l}\text{source dependent}\\ \text{pair}\end{array}$$

The first pair is called tautologous because these two equations are <u>automatically</u> true if $\vec{E}$ and $\vec{B}$ come from scalar and vector potentials:

$$\vec{E} = \nabla\phi + \frac{\partial\vec{A}}{\partial t}$$
$$\vec{B} = \nabla\times\vec{A}$$

E&M is usually done with the assumption that spacetime is <u>continuous</u>. We want to understand E&M on a <u>discrete</u> version of spacetime — a hypercube lattice!

Discrete Electric & Magnetic Fields

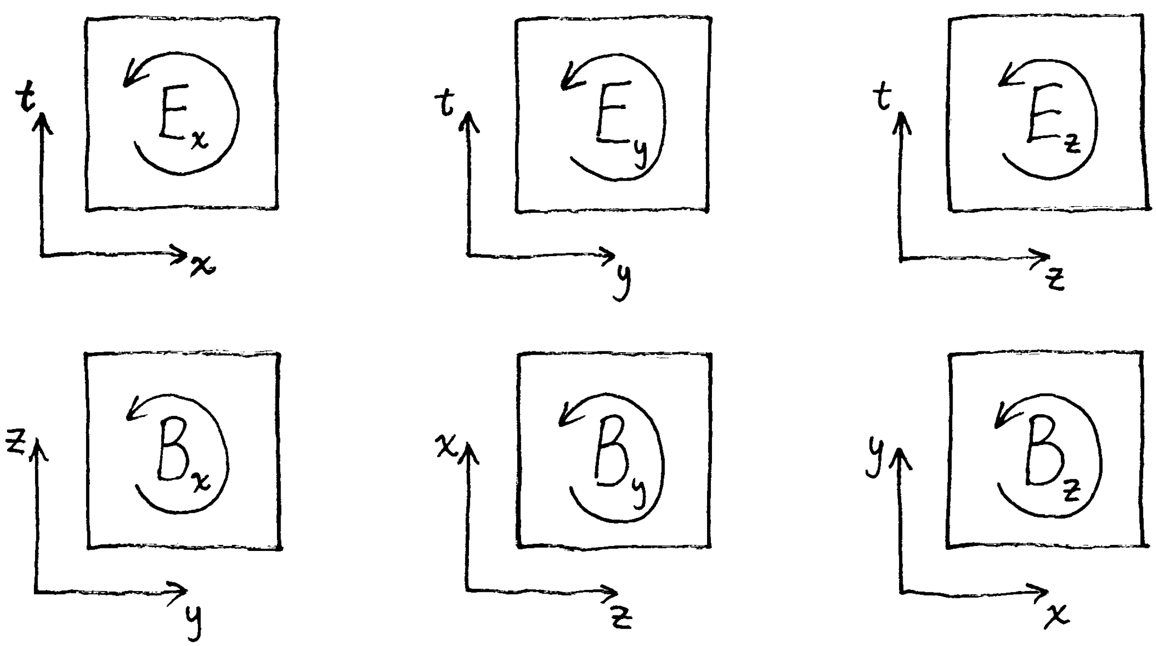
A hypercube lattice has 2-dimensional faces oriented in <u>six</u> different ways. Namely, it has faces parallel to each of the planes:

$$xt \qquad yt \qquad zt \qquad yz \qquad zx \qquad xy$$

On each of these types of faces lives one of the <u>six</u> components of the electromagnetic field: respectively,

$$E_x \qquad E_y \qquad E_z \qquad B_x \qquad B_y \qquad B_z$$

The <u>electromagnetic field</u> on a hypercube lattice thus labels each face by a number $E_x, E_y, \ldots,$ or B_z according to its orientation:

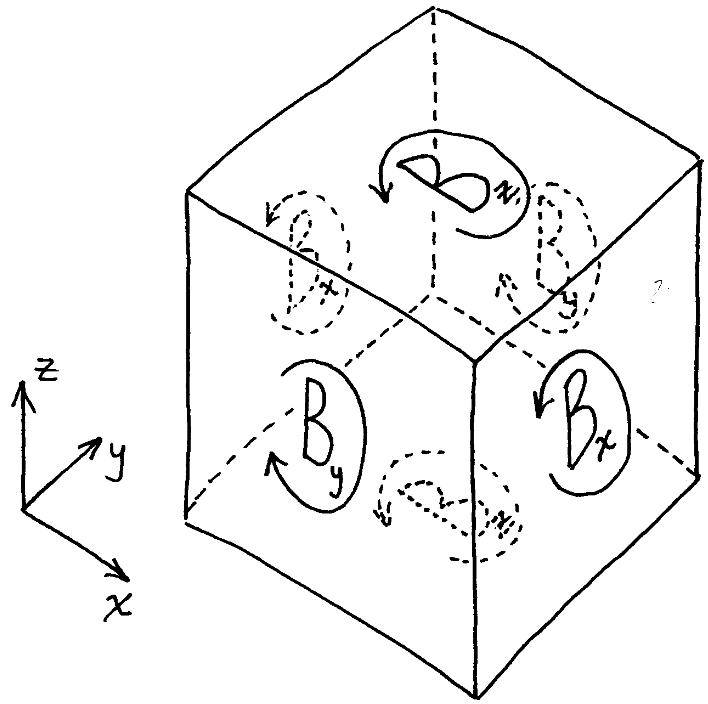


<u>Maxwell's equations</u> on this lattice version of spacetime give us rules for how we are allowed to pick the values of $E_x, E_y, \ldots, B_z$ on each face!

The Source Independent Pair of Maxwell's Equations

1) $\nabla \cdot \vec{B} = 0$ says $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$. On a discrete spacetime this becomes $\frac{\Delta B_x}{\Delta x} + \frac{\Delta B_y}{\Delta y} + \frac{\Delta B_z}{\Delta z} = 0$, or since $\Delta x = \Delta y = \Delta z = \Delta t$ for a hypercube:

$$\boxed{\Delta B_x + \Delta B_y + \Delta B_z = 0}$$



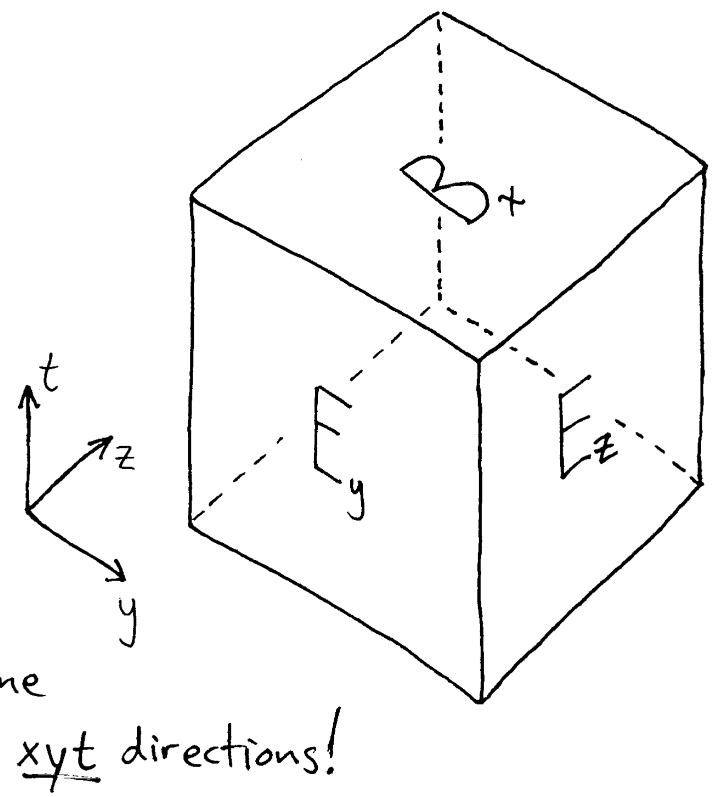
This is all about a cubical 3-face in the <u>xyz</u> direction. It says "the 'oriented sum' around this 3-face is zero."

Similarly...

2) $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ has three components. The first one is $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} = 0$

or on the lattice:

$$\boxed{\Delta E_z - \Delta E_y + \Delta B_x = 0}$$



which says "the oriented sum of the electromagnetic field around a 3-face in the <u>yzt</u> direction is zero"

The other components give the same result for 3-faces in <u>xzt</u> and <u>xyt</u> directions!

Summary:

The electromagnetic field assigns to each square face in our spacetime lattice a number. The two source-independent Maxwell equations can be stated verbally as follows:

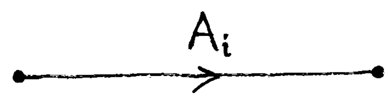
> "The oriented sum of the electromagnetic field around any 3-cube in spacetime is zero."

The other two Maxwell equations give additional conditions the field must satisfy, but for now, let's stick to the source-independent equations and see how we can choose labels on faces so these are satisfied.

We will see the source independent pair is automatically true if the electromagnetic field comes from a "potential"! ...

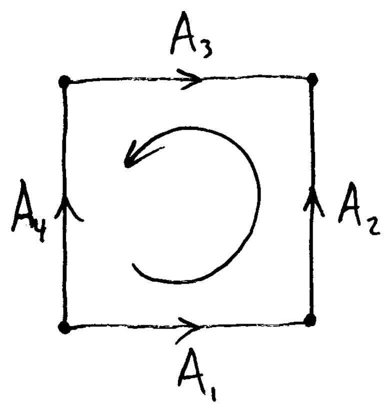
The Potential

To each edge e_i in the spacetime lattice, the <u>potential</u> A assigns a number A_i



To construct the electromagnetic field from the potential, we label each face by the 'oriented sum' of the potential around its boundary:

This face:



gets labelled by the number

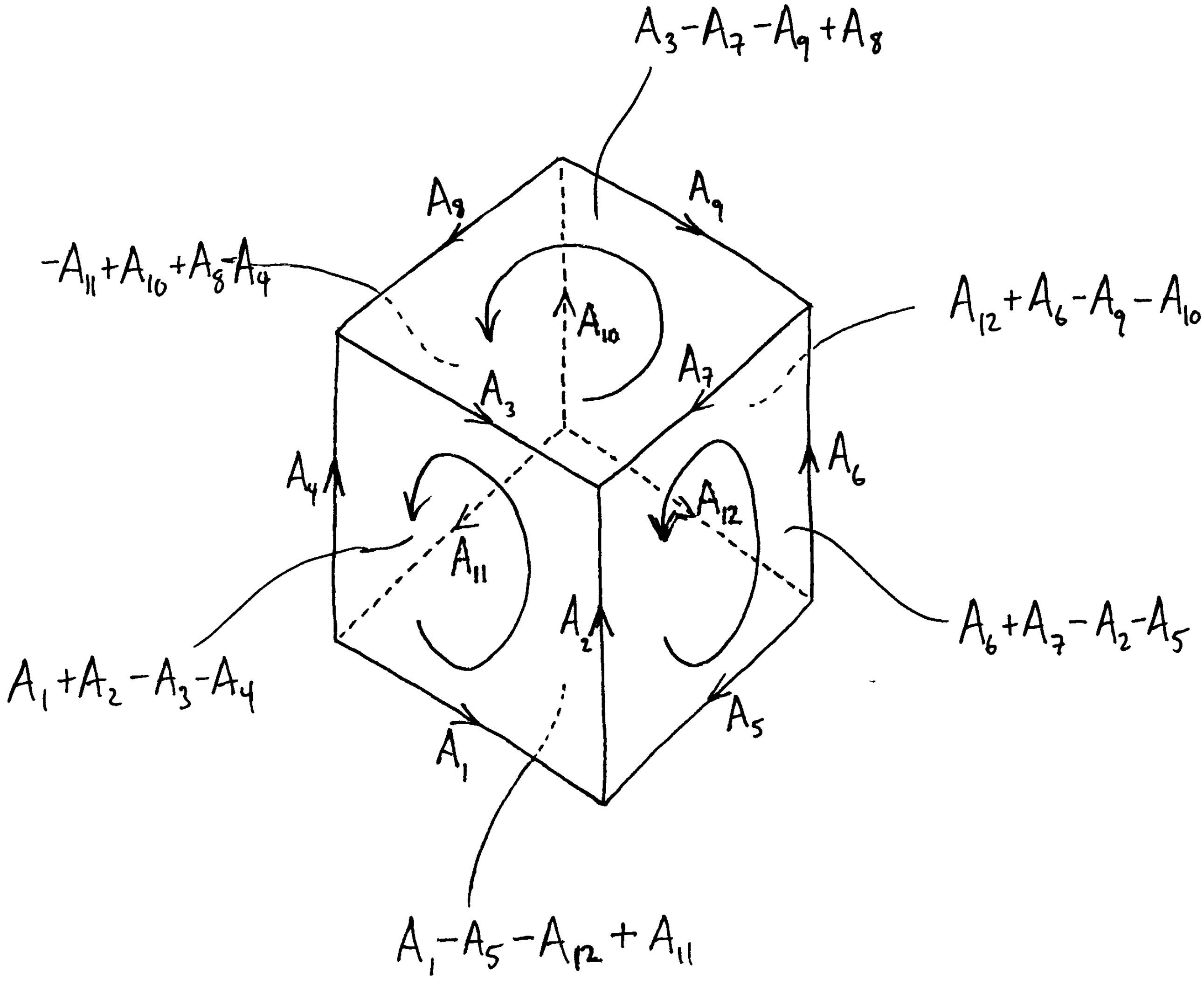
$$A_1 + A_2 - A_3 - A_4$$

For example, if this face happened to be parallel to the xt plane we would say

$$E_x = A_1 + A_2 - A_3 - A_4$$

on this face.

When the electromagnetic field comes from a potential, the source independent pair of Maxwell's equations is automatically satisfied, since everything cancels in pairs:



Oriented sum of face labels

= (sum of front faces) − (sum of back faces)

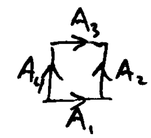
$$= (A_1 + A_2 - A_3 - A_4 + A_6 + A_7 - A_2 - A_5 + A_3 - A_7 - A_9 + A_8)$$
$$- (A_1 - A_5 - A_{12} + A_{11} - A_{11} + A_{10} + A_8 - A_4 + A_{12} + A_6 - A_9 - A_{10})$$

$= 0$ "The oriented sum of an oriented sum is zero"

Finishing Up

- Review:

  Electromagnetic field is just numbers labelling faces $\boxed{E_x}$

  Potential is just numbers labelling edges 

  Two of Maxwell's equations are tautologous

- The other two Maxwell's equations come from extra conditions demanded of the potential by the sources (charge and current)

- One can check that in the limit as the lattice spacing shrinks to zero, one recovers the theory of electromagnetism in the continuum.