1. Introduction and statement of the results.

The uniqueness in the Cauchy problem and the connected unique continuation property (ucp) for subelliptic operators is a subject which is far from being understood and to a large extent unexplored. On the negative side there exists a general counterexample of Bahouri [Ba] to the ucp for zero order perturbations of sub-Laplacians \( \mathcal{L} - V = \sum_{j=1}^{m} X_j X_j - V \), when, besides the finite rank condition on the Lie algebra, some additional geometric conditions are fulfilled by the vector fields \( X_1, \ldots, X_m \) (such additional assumptions are not necessary in dimension three or four). What happens, however, if one considers the unperturbed operator corresponding to the case \( V = 0 \)? In this situation Bony [Bo] has proved uniqueness in the Cauchy problem if the vector fields are real analytic. A general satisfactory answer to this question in the \( C^\infty \) or less regular case does not seem to be presently available. In this paper we study the strong unique continuation property (sucp) for a class of “variable coefficient” operators whose “constant coefficient” model at one point is the so called Baouendi-Grushin operator [B], [Gr1], [Gr2]. We recall that the latter is the following operator on \( \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n \), \( N = n + m \),

\[
\mathcal{L}_o = \sum_{i=1}^{N} X_i X_i u,
\]

where the vector fields are given by

\[
X_k = \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n, \quad X_{n+j} = |x|^{\alpha} \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, m.
\]

Here \( \alpha > 0 \) is a fixed parameter, \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, \ldots, y_m) \in \mathbb{R}^m \). When \( \alpha = 0 \), \( \mathcal{L}_o \) is just the standard Laplacian in \( \mathbb{R}^N \). For \( \alpha > 0 \) the ellipticity of the operator \( \mathcal{L}_o \) becomes degenerate on the characteristic submanifold \( M = \mathbb{R}^n \times \{0\} \) of \( \mathbb{R}^N \). The analysis of the operator \( \mathcal{L}_o \) is very subtle and closely connected to that of the real part of the Kohn sub-Laplacian on the Heisenberg group \( \mathbb{H}^n \), see [RS], [G2]. The latter operator is real-analytic hypoelliptic, thus harmonic functions in \( \mathbb{H}^n \) cannot vanish to infinite order at one point unless they are identically zero. However, to present date there exists no quantitative proof of such sucp in \( \mathbb{H}^n \). In particular, it would be important to know whether the generalized frequency introduced in [GLa] is increasing, but this remains at the moment a challenging
open question. Such and related questions constitute the background motivation of the present paper. Returning to the operator $\mathcal{L}_o$, we mention that it was proved in [G2] that the frequency attached to the horizontal energy is indeed increasing at points of the degeneracy manifold $M$, thus the sucp holds for $\mathcal{L}_o$. In the same paper this is also proved for the operator $\mathcal{L}_o - \overrightarrow{b} \cdot Du - V$ with suitable assumptions on $\overrightarrow{b}$ and $V$. To give an idea, for example

$$|V| \leq \frac{C}{\rho} \psi \quad \text{and} \quad |\overrightarrow{b} \cdot Du| \leq C|Xu| \psi^{1/2}$$

is enough. Here $Du$ is the gradient of $u$, $|Xu|$ is the horizontal gradient (1.8) of $u$, and $\rho$ and $\psi$ are defined correspondingly in (1.6) and (1.7). With a completely different method, based on a subtle two-weighted Carleman estimate, the sucp was established in [GS1] for zero order perturbations $\mathcal{L}_o - V$, where the potential $V$ is allowed to belong to some appropriate $L^p$ spaces.

In this paper we consider equations of the type

$$\mathcal{L}u = \sum_{i,j=1}^{N} X_j(a_{ij}(x,y))X_iu = 0. \tag{1.3}$$

We assume that $A = (a_{ij}(x,y))$, $i,j = 1, \ldots, N$, is a $N \times N$ matrix-valued function on $\mathbb{R}^N$ which, for simplicity, we take such that

$$A(0) = Id. \tag{1.4}$$

Furthermore, we assume $A$ is symmetric and uniformly elliptic matrix. Thus $a_{ij}(g) = a_{ji}(g)$ and there exists $\lambda > 0$ such that for any $\eta \in \mathbb{R}^N$

$$\lambda|\eta|^2 \leq < A\eta, \eta > \leq \lambda^{-1}|\eta|^2. \tag{1.5}$$

Our main concern is whether, under suitable assumptions on the matrix $A$, the sucp continues to hold for the operator $\mathcal{L}$. To put our result in perspective we mention that when $\alpha = 0$ in (1.2), so that $\mathcal{L}_o$ is the standard Laplacian, a famous result due to Aronszajn, Krzywicki and Szarski [AKS] states that if the matrix $A$ has Lipschitz continuous coefficients, then the operator $\mathcal{L}$ possesses the sucp. Furthermore, it was shown in [M] that such assumption is optimal. Our results, Theorems 1.2 and 1.3 can be seen as a generalization of that in [AKS], in the sense that, in the limit as $\alpha \to 0$ we recapture both the assumptions and the conclusion of the elliptic case, see Remark 1.3. The approach, however, is different from that in [AKS], which is based on Carleman inequalities along with results from Riemannian geometry that do not seem to be adaptable to our context due to the lack of ellipticity. Instead, we have borrowed the ideas developed in [GL1], [GL2], [G2], see also the subsequent simplification in [K]. Our main result is Theorem 1.2, which gives a quantitative control of the order of zero of a weak solution to (1.3). Such result is proved under some hypothesis on the matrix $A$ which are listed as assumptions (H) below. The latter are tailor made on the geometry of the operator $\mathcal{L}_o$ and should be interpreted as a sort of Lipschitz continuity with respect to a suitable pseudo-distance associated to the system of vector fields (1.2).
In order to state the main result we recall the definition of the gauge $\rho$ associated to $L_o$ [G2] with $\xi = (x, y)$.

$$\rho = \rho(\xi) \overset{df}{=} (|x|^{2(\alpha+1)} + (\alpha + 1)^2|y|^2)^{1/2(\alpha+1)}.$$  

In the sequel we indicate with $B_r = \{\rho < r\}$ the pseudo-balls with respect to $\rho$ centered at the origin in $\mathbb{R}^N$ with radius $r$. The outer unit normal on $\partial B_r$ is given by $\nu = |D\rho|^{-1}D\rho$. It is worth stressing that if $\alpha = 2k$, with $k \in \mathbb{N}$, then the the system of vector fields in (1.2) satisfies H"{o}rmander’s condition, and the ensuing Carnot-Carathéodory distance is comparable to $\rho(\xi)$. We will also need the angle function $\psi$ defined as follows [G2]

$$\psi = \psi(\xi) \overset{df}{=} |X\rho|^2(\xi) = \frac{|x|^{2\alpha}}{\rho^{2\alpha}}, \quad \xi \neq 0.$$  

Hereafter, given a function $f$, we denote by

$$Xf = (X_1f, \ldots, X_Nf)$$  

the gradient along the system of vector fields in (1.2) (called also horizontal gradient of $f$), and let $|Xf|^2 = \sum_{j=1}^N(X_jf)^2$. The function $\psi$ vanishes at every point of the characteristic manifold $M$, and clearly satisfies $0 \leq \psi \leq 1$.

**Definition 1.1.** A weak solution to $Lu = 0$ in an open set $\Omega$ is a function $u \in C(\Omega)$ such that the (distributional) horizontal gradient $Xu \in L^2_{\text{loc}}(\Omega)$ and the equation $Lu = 0$ is satisfied in the variational sense in $\Omega$, i.e.,

$$\int_{\Omega} \langle AXu, X\phi \rangle dV = 0$$

for every $\phi \in C_0^\infty(\Omega)$.

For convenience, we have required that a weak solution be a continuous function since we will take the trace of the latter on hypersurfaces. We note however that such assumption could be considerably relaxed if one assumes the existence of sub-unit curve joining any two points. Under this additional hypothesis, the assumption $u, Xu \in L^2_{\text{loc}}(\Omega)$ would suffice to apply the results in [FL], [FS], and conclude that a weak solution $u$ is (after modification on a set of measure zero) Hölder continuous with respect to the Carnot-Carathéodory distance, and therefore (with a different exponent) also with respect to the Euclidean distance. Of course, when $\alpha = 2k$ we have already mentioned that vector fields $X_i$ satisfy the Hörmander finite rank condition, thus the existence of a sub-unit curve joining any two points follows from the theorem of Chow-Rashevsky. We are ready to state our main result.

**Theorem 1.2.** Let $A$ be a symmetric matrix satisfying (1.5) and the hypothesis (H) below with relative constant $\Lambda$. Suppose $u$ is a weak solution of (1.3) in a neighborhood of the origin $\Omega$. Under these assumptions, there exist positive constants $C = C(u, \alpha, \lambda, \Lambda, N)$ and $r_o = r_o(u, \alpha, \lambda, \Lambda, N)$, such that, for any $2r \leq r_o$, we have

$$\int_{B_{2r}} u^2 \psi \, dV \leq C \int_{B_r} u^2 \psi \, dV.$$
The dependence of the constant $C$ on $u$ is quite explicit. It involves the $L^2$ norm of $|Xu|$ on $B_1$, and the $L^2$ norm of $u$ on $\partial B_1$ with respect to the weighted measure $\psi \, dH_{N-1}$. We remark that, although we have stated Theorem 1.2 when the point of consideration is the origin, this result continues to be true for any other point with the appropriate modification of the hypothesis (H).

We say that $u \in L^2_{\text{loc}}(\mathbb{R}^N)$ vanishes to infinite order at some $z_0 \in \mathbb{R}^N$ if for every $k > 0$ one has

$$
\lim_{r \to 0} \frac{1}{r^k} \int_{B_r(z_0)} |u(z)|^2 \, dV = 0.
$$

A given partial differential operator $\mathcal{L}$ in $\mathbb{R}^N$ is said to possess the strong unique continuation property (SUCP) if for every $z_0 \in \mathbb{R}^N$, and any weak solution $u$ of $\mathcal{L}u = 0$, the assumption that $u$ vanishes to infinite order at $z_0$ implies that $u \equiv 0$ in some neighborhood of $z_0$. In other words non-trivial solutions can have at most finite order of vanishing. As it is well known [GL1], Theorem 1.2 implies the following sucp.

**Theorem 1.3.** With the assumptions of Theorem 1.2, the operator $\mathcal{L}$ has the SUCP.

In order to state our main assumptions (H) on the matrix $A$ it will be useful to think of the latter in the following block form, $A_{12} = A_{21}:

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
$$

Here, the entries are respectively $n \times n$, $n \times m$, $m \times n$ and $m \times m$ matrices. We shall denote by $B$ the matrix

$$
B = A - I_{N \times N}
$$

and thus

$$
(1.9) \quad B(0) = O_{N \times N},
$$

thanks to (1.4). The proof of Theorem 1.2 relies crucially on the following assumptions on the matrix $A$. These will be our main hypothesis and, without further mention, will be assumed to hold throughout the paper.

**HYPOTHESIS.** There exists a positive constant $\Lambda$ such that, for some $\epsilon > 0$, one has in $B_r$ the following estimates

$$
|b_{ij}| = |a_{ij} - \delta_{ij}| \leq \begin{cases} 
\Lambda \rho, & \text{for } 1 \leq i, j \leq n \\
\Lambda \psi^{1+\frac{1}{2\alpha}} \rho = \Lambda \frac{|x|^{\alpha+1}}{\rho}, & \text{else}
\end{cases}
$$

(H)

$$
|X_k b_{ij}| = |X_k a_{ij}| \leq \begin{cases} 
\Lambda, & \text{for } 1 \leq k \leq n, \text{ and } 1 \leq i, j \leq n \\
\Lambda \psi^{\frac{1}{2}} = \Lambda \frac{|x|^\rho}{\rho}, & \text{else}
\end{cases}
$$

\[ \]

\[ \]
An interesting, typical example of a matrix satisfying the conditions (H) is

\[
A = \begin{pmatrix}
1 + \rho f(x, y) & |x|^{\alpha+1} g(x, y) \\
|x|^{\alpha+1} g(x, y) & 1 + |x|^{\alpha+1} h(x, y)
\end{pmatrix},
\]

where \( f, g \) and \( h \) are functions which are Lipschitz continuous at the origin of \( \mathbb{R}^2 \) with respect to the Euclidean metric. In this example \( m = n = 1 \).

**Remark 1.4.** It is important to observe that, thanks to (1.7), if we take formally \( \alpha = 0 \) in (H) we obtain a Lipschitz condition at the origin for the matrix \( A \). Our results thus encompass those in the cited paper [AKS], see also [GL1].

For a vector field \( F \) we denote by \( FA \) the matrix with elements \( (F a_{ij}) \). We will apply the same notation to all matrices under consideration. Throughout the paper we will tacitly assume that all vectors are column vectors. Also, we will use the same notation for first order partial differential operators and for the corresponding tangent vectors, with meaning determined by the context.

The plan of the paper is as follows. In section two we prove Theorem 1.2. The proof involves various technical estimates. For the reader’s convenience and ease of exposition we have collected all the auxiliary material in section three.

## 2. The frequency function.

The purpose of this section is to prove Theorem 1.2. The main step is to show the monotonicity of the frequency Theorem 2.2. We begin by introducing the relevant quantities that will appear in the proof. Since our results are local in nature, from now on, we focus our attention on the pseudo-ball \( B_2 \). The notation \( dH_{N-1} \) will indicate \((N-1)\)-dimensional Hausdorff measure in \( \mathbb{R}^N \). Let \( u \) be a weak solution of (1.3) in \( B_2 \).

**Definition 2.1.** For every \( 0 < r < 2 \) we let

\[
H(r) = \int_{\partial B_r} u^2 \frac{< AX \rho, X \rho >}{|D \rho|} dH_{N-1},
\]

\[
D(r) = \int_{B_r} < AX u, X u > dV.
\]

The generalized frequency of \( u \) on \( B_r \) is defined by

\[
N(r) \overset{\text{def}}{=} \begin{cases} 
\frac{r D(r)}{H(r)}, & \text{if } H \neq 0 \\
0, & \text{if } H = 0.
\end{cases}
\]

We shall denote by \( S \) the matrix relating the gradient along the vector fields in (1.2) and the standard gradient in \( \mathbb{R}^N \), i.e., \( X = S D \), where

\[
S = \begin{pmatrix}
I_{n \times n} & 0 \\
0 & |x|^{\alpha} I_{m \times m}
\end{pmatrix}.
\]
Trivially, we have
\begin{equation}
S = S^t \quad \text{and} \quad \mathcal{L}u = \text{div}(SASDu).
\end{equation}

The following theorem constitutes the main result of this section.

**Theorem 2.2.** Let \( u \) be a nontrivial weak solution of \( \mathcal{L}u = 0 \) in the pseudo-ball \( B_2 \), then there exist positive constants \( r_o = r_o(\alpha, \lambda, \Lambda, N) \) and \( M = M(u, \alpha, \lambda, \Lambda, N) \) such that
\[
\tilde{N}(r) = \exp(Mr)N(r)
\]

is a continuous monotonically nondecreasing function for \( r \in (0, r_o) \).

**Proof.** The proof of Theorem 2.2 rests on Lemmas 2.5 and 2.12 below. Let \( M = \max\{C_1, C_2\} \), where \( C_1 \) and \( C_2 \) are the constants from Lemmas 2.5 and 2.12. Let \( Q \) be the homogeneous dimension in (2.3) associated with the non-isotropic dilations (2.4). With \( r_o \) as defined in Lemma 2.5 we have that, either \( u \equiv 0 \) in \( B_{r_o} \), or \( H(r) > 0 \) for \( 0 < r < r_o \). In the former case the frequency is identically zero on \( (0, r_o) \), so let us consider the latter case, in which \( H(r) > 0 \). The continuity of \( \tilde{N}(r) \) follows from the continuity of each of the functions involved in its definition. Furthermore, for a.e. \( r \in (0, r_o) \) we have
\[
\left( \ln \frac{rD(r)}{H(r)} e^{2Mr} \right)' = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} + 2M \\
\geq \frac{1}{r} + \frac{Q - 2}{r} + \frac{2}{D(r)} \int_{\partial B_r} <AXu, X\rho >^2 dH_{N-1} \\
- \frac{Q - 1}{r} - 2\frac{D(r)}{H(r)} \geq 0,
\]
where we have applied first Lemmas 2.5 and 2.12, and then Proposition 2.4 and the Cauchy-Schwarz inequality.

With the help of the monotonicity it is easy to prove Theorem 1.2, see Section 3 of [GL1]. We include the proof in the current setting for completeness.

**Proof of Theorem 1.2.** If the solution vanishes in some neighborhood of the origin then the doubling for all sufficiently small balls is trivially satisfied. Let us consider next the case of a non-trivial solution. Let \( r_o \) be the number defined in Lemma 2.5 and \( 2r \leq r_o \). By the co-area formula
\[
\int_0^R \int_{\partial B_r} u^2 \phi \frac{dH_{N-1}}{|D\rho|} dr = \int_{B_R} u^2 \phi dV.
\]
From the ellipticity of \( A \) in (1.5), we have
\[
\int_0^R H(r) dr \approx \int_{B_R} u^2 \phi dV,
\]
with constant of proportionality depending only on $\lambda > 0$. This shows it is enough to prove the doubling property for the height function $H$. Now, we obtain from Lemma 2.5
\[
\ln \frac{H(2r)}{2^Q-1 H(r)} = \ln \frac{H(2r)}{2^Q-1 r^Q} - \ln \frac{H(r)}{r^Q} = \int_r^{2r} \left\{ \frac{H'(t)}{H(t)} \frac{t}{Q} - \frac{Q-1}{t} \right\} dt \\
\leq \int_r^{2r} \left\{ 2 \frac{D(t)}{H(t)} + C_1 \right\} dt \leq \int_r^{2r} 2\tilde{N}(t) \frac{e^{-2Mt}}{t} dt + Mr \\
\leq 2\tilde{N}(r_o) \int_r^{2r} \frac{1}{t} dt + M = 2\tilde{N}(r_o) \ln 2 + M,
\]
where in the last inequality we have used the monotonicity of the modified frequency expressed by Theorem 2.2. We thus conclude
\[
H(2r) \leq 2^{Q-1} e^{\{2\tilde{N}(r_o) \ln 2 + M\}} H(r).
\]
Integrating the latter inequality we obtain the doubling property in the conclusion of Theorem 1.2.

\[\square\]

\textbf{Remark 2.3.} We observe that for non-trivial solution we have the doubling property for all balls $B_{2r} \subset \Omega$ and $2r \leq 1$, since for "big" balls, i.e., $2r \geq r_o$ we have
\[
\int_{B_{2r}} u^2 \psi dV \leq \int_{B_r} u^2 \psi dV.
\]
Of course, in this case the constant $C$ in the doubling property depends on $\tilde{N}(1)$.

Finally, we establish Theorem 1.3.

\textbf{Proof of Theorem 1.3.} Suppose $u$ is a solution which vanishes to infinite order at the origin. Let $|B_r| = \omega_o r^Q$. Fix a number $\kappa > 0$ such that $C_o 2^{-Q\kappa} = 1$. For any $r$ sufficiently small and $p \in \mathbb{N}$ the doubling property applied $p$ times gives
\[
\int_{B_r} u^2 \psi dV \leq C_0^p \int_{B_{r/2^p}} u^2 \psi dV \\
\leq \omega_o^p C_0^p r^{Q\kappa} \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2 \psi dV \\
\leq \omega_o^p C_0^p Q\kappa \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2 \psi dV \to 0
\]
when $p \to \infty$ since $0 \leq \psi \leq 1$. This ends the proof.
\[\square\]

The remainder of this section is devoted to establishing Lemmas 2.5 and 2.12.
Proposition 2.4. For a.e. \( r \in (0, 2) \) the horizontal energy of \( u \) on \( B_r \) can be expressed by the surface integral

\[
D(r) = \int_{\partial B_r} u \frac{\langle AXu, X\rho \rangle}{|D\rho|} dH_{N-1}.
\]

Proof. By the definition of weak solution we have \( u \) is continuous and \( Xu \in L^2(B_2) \), thus for a.e. \( r \in (0, 2) \) one has \( Xu \in L^2(\partial B_r) \). The outer unit normal on \( \partial B_r \) is given by

\[
\nu = \frac{D\rho}{|D\rho|} - D\rho.
\]

The divergence theorem, (2.2) and the fact that \( Lu = 0 \) imply

\[
\int_{\partial B_r} u \frac{\langle AXu, X\rho \rangle}{|D\rho|} dH_{N-1} = \int_{B_r} \text{div}(uSA Xu) dV = \int_{B_r} \langle AXu, Xu \rangle dV + \int_{B_r} Lu dV
\]

as claimed in the proposition. \( \square \)

We proceed with proving the main estimate for the generalized height function \( H(r) \). This is the first place where the assumptions (H) on the matrix \( A \) play a decisive role. We observe that \( r \to H(r) \) is absolutely continuous, thus differentiable a.e. on \((0, 2)\). In the subsequent analysis the number

\[
Q = n + (\alpha + 1)m \quad (> N = n + m),
\]

will play an important role. We note that \( Q \) is the homogeneous dimension relative to the anisotropic dilations

\[
\delta_t(\xi) = \delta_t(x, y) = (tx, t^{(\alpha+1)}y), \quad t > 0
\]

naturally associated with the vector fields in (1.2). The infinitesimal generator of (2.4) is

\[
Z = \sum_{1 \leq i \leq n} x_i \frac{\partial}{\partial x_i} + (\alpha + 1) \sum_{1 \leq j \leq m} y_i \frac{\partial}{\partial y_i},
\]

so that a function \( u \) is \( \delta_t \)-homogeneous of degree \( k \in \mathbb{R} \) if and only if \( Zu = ku \). At this point it is worth observing that if \( u \) is homogeneous of degree \( k \), and solves the ”constant coefficient” equation \( L_\phi u = 0 \) (i.e. \( u \) is a fundamental \( L_\phi \)-harmonic of degree \( k \)), then the corresponding frequency is constant and equal to \( k \). This justifies the name generalized frequency. To prove this fact one uses Proposition 2.4 with \( A \equiv I \) which gives
\[
D(r) = \int_{B_r} < Xu, Xu > dV = \int_{\partial B_r} u < Xu, X\rho > \frac{dH_{N-1}}{|D\rho|}.
\]

A calculation, see (2.13) in [G2] or Proposition 3.1, shows (\( X = SD \! )

\[
(2.6) \quad X\rho = \frac{\psi}{\rho} S^{-1}Z,
\]

for any function \( u \). When \( u \) is \( \mathcal{L}_\omega \)-harmonic of degree \( k \) we have \( Zu = ku \), and one infers from (2.6)

\[
<Xu, X\rho > = \frac{\psi}{\rho} Zu.
\]

Substitution of the latter identity in (2.6) gives

\[
D(r) = k \int_{\partial B_r} u^2 \frac{\psi}{|D\rho|} dH_{N-1} = \frac{k}{r} H(r),
\]

which proves \( N(r) \equiv k \).

**Lemma 2.5.** a) There exists a positive constant \( C_1 = C_1(\alpha, \lambda, \Lambda, N) \) such that for a.e. \( r \in (0,2) \) one has

\[
\left| H'(r) - \frac{Q-1}{r} H(r) - 2D(r) \right| \leq C_1 H(r).
\]

b) There exists a positive number \( r_o = r_o(\alpha, \lambda, \Lambda, N) \leq 1 \) such that, either \( H(r) = 0 \) on \( (0,r_o) \), or \( H(r) > 0 \) on \( (0,r_o) \).

**Proof.** a) Using the definition (2.1) of \( S \) we have

\[
\frac{< AX\rho, X\rho >}{|D\rho|} = < SAX\rho, \nu >.
\]

The divergence theorem gives

\[
H(r) = \int_{\partial B_r} u^2 < SAX\rho, \nu > dH_{N-1} = \int_{B_r} \text{div}(u^2 SAX\rho) dV
\]

\[
= \int_{B_r} < AX\rho, Xu^2 > dV + \int_{B_r} u^2 \mathcal{L}\rho dV
\]

\[
= \int_{B_r} 2u < AX\rho, Xu > dV + \int_{B_r} u^2 \mathcal{L}\rho dV.
\]

Since the gauge \( \rho \) is not smooth at the origin, to make rigorous the previous calculation one must integrate on the set \( B_r \setminus \overline{B}_\varepsilon \) and then let \( \varepsilon \to 0 \). We note that the last integral on the
second line of the above chain of equalities is convergent since \( \mathcal{L}_\rho \in L^1_{\text{loc}}(\mathbb{R}^N) \). This can be seen from the remarkable formula

\[
\mathcal{L}_\rho \rho = \frac{Q-1}{\rho} |X\rho|^2, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

which is (2.18) in [G2]. Once (2.8) is available one easily obtains by a rescaling, using (2.4), that \( \rho^{-p} \in L^1_{\text{loc}}(\mathbb{R}^N) \) if and only if \( p < Q \). This shows, in particular, that \( \mathcal{L}_\rho \rho \in L^1_{\text{loc}}(\mathbb{R}^N) \).

We note explicitly that (2.8) expresses, in disguise, the fact that for a suitable constant \( C > 0 \) the function

\[
\Gamma = C \rho^{2-Q}
\]

is a fundamental solution of \( \mathcal{L}_\rho \) with pole at 0.

Returning to (2.7), after an application of the Federer’s co-area formula we differentiate at a.e. \( r > 0 \), and use Proposition 2.4, obtaining

\[
H'(r) = 2D(r) + \int_{\partial B_r} \frac{u^2 L_\rho}{|D\rho|} dH_{N-1}.
\]

This implies

\[
H'(r) - \frac{Q-1}{r} H(r) - 2D(r) = \int_{\partial B_r} u^2 \frac{\mathcal{L}_\rho}{|D\rho|} dH_{N-1} - \frac{Q-1}{r} H(r)
\]

\[
= \int_{\partial B_r} u^2 \frac{\text{div}(SBX\rho)}{|D\rho|} dH_{N-1} + \int_{\partial B_r} u^2 \frac{\mathcal{L}_0 \rho}{|D\rho|} dH_{N-1}
\]

\[
- \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{|X\rho|^2}{|D\rho|} dH_{N-1}
\]

\[
- \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{<BX\rho, X\rho>}{|D\rho|} dH_{N-1}.
\]

We recall that \((b_{ij}) = B = A - Id\). Now, thanks to (2.8) the two middle terms in the last equality above are equal. The last term is easily estimated as follows on \( \partial B_r \),

\[
\frac{<BX\rho, X\rho>}{|D\rho|} \leq C r \frac{AX\rho, X\rho}{|D\rho|},
\]

for some positive constant \( C = C(\alpha, \lambda, \Lambda, N) \). This is recognized observing that by (H) we have \( ||B||_{L^\infty(\partial B_r)} \leq Cr \), and using also (1.5). Finally, we estimate the first term in the right-hand side. Writing the divergence term as

\[
\text{div}(SBX\rho) = \sum_{i,j=1}^{N} X_i (b_{ij} X_j \rho) = \sum_{i,j=1}^{N} X_i b_{ij} X_j \rho + b_{ij} X_i X_j \rho,
\]
and taking into account the assumptions (H), Proposition 3.1 and Proposition 3.3 we find, by splitting the terms into the four groups that appear in the block form of \( A \) (and hence of \( B \)), the following inequalities

\[
\sum_{i,j=1}^{N} |X_i b_{ij} X_j \rho| \leq C\left(\psi^{1+1/n} + \psi^{1/2} \psi^{1+1/n} + \psi^{1/2} \psi^{1/2} + \psi^{1/2} \psi^{1/2}\right) \leq C\psi,
\]

\[
\sum_{i,j=1}^{N} |b_{ij} X_i X_j \rho| \leq C\left(\rho \psi^{1/2} + \rho \psi^{1/2} + \rho \psi^{1/2} \psi^{1/2} + \rho \psi^{1/2} \psi^{1/2}\right) \leq C\psi.
\]

This completes the proof of part a).

b) From part a) we have

\[
H'(r) \geq \left(\frac{Q-1}{r} - C_1\right) H(r) + 2D(r).
\]

Let \( r_1 = \min\{1, \frac{Q-1}{2C_1}\} \) so that \( H'(r) \geq C_1 H(r) + 2D(r) \geq 0 \) on the interval \((0, r_1)\). Therefore there exists an \( 0 < r_o \leq r_1 \) with the required properties. \( \square \)

Our next objective is to obtain estimates of the first variation \( D'(r) \) of the horizontal energy. Let

\[
(2.10) \quad \mu \overset{\text{def}}{=} \langle AX \rho, X \rho \rangle.
\]

Consider the vector field \( F \) defined as follows

\[
(2.11) \quad F = \rho \sum_{i,j=1}^{N} \frac{a_{ij} X_j \rho}{\mu} X_i, \quad x \neq 0,
\]

i.e.

\[
F u = \frac{\rho}{\mu} \langle AX \rho, X u \rangle = \frac{\rho}{\mu} \langle SAX \rho, Du \rangle,
\]

for any smooth function \( u \). We now see that the assumptions on the matrix \( A \) guarantee that \( F \) can be continuously extended to all of \( \mathbb{R}^N \). Furthermore, near the characteristic manifold, such extension gives a small perturbation of the Euler vector field \( Z \) in (2.5). To prove this latter claim, we recall (2.6), and let

\[
(2.12) \quad \sigma \overset{\text{def}}{=} \langle BX \rho, X \rho \rangle = \mu - \psi.
\]

Thus, \( F \) can be re-written as

\[
(2.13) \quad F = \frac{\psi}{\mu} Z + \frac{\rho}{\mu} SBX \rho = Z - \frac{\sigma}{\mu} Z + \rho \sum_{i,j=1}^{N} \frac{b_{ij} X_j \rho}{\mu} X_i.
\]
From (H), the coercivity of \( A \), and from Lemma 3.1 we find easily

\[
\left| \sigma \right| \leq C \rho \psi^{1+\frac{1}{2}} \left| Z \right|
\]

(2.14)

and

\[
\left| b_{ij} X_j \rho X_i \right| \leq C \rho \psi^{1+\frac{1}{2}} \leq C |x|.
\]

Substituting the two estimates (2.14) in (2.13), we obtain the above claim.

Our next goal is establishing a basic Rellich-type identity involving the vector field \( F \), Lemma 2.11, which we shall use to prove the main estimate on the derivative of the horizontal energy, see Lemma 2.12. The proof of such Rellich-type identity relies on some basic estimates on the divergence and the commutators of \( F \) which are collected in the subsequent Lemmas 2.6, 2.7, 2.8, 2.9 and 2.10. We mention that, in turn, the proofs of these five lemmas rely on some auxiliary technical estimates which, in order to keep the flow of this section, we have collected separately in the next section. Hereafter, the summation convention over repeated indices will be adopted.

**Lemma 2.6.** There exists a constant \( C = C(\alpha, \lambda, \Lambda, N) > 0 \) such that for \( 1 \leq i \leq N \) we have:

\[
\left| [X_i, \frac{\rho}{\mu} S B X \rho] u \right| \leq C \rho |X u|.
\]

**Proof.** By a direct calculation

\[
[X_i, \frac{\rho}{\mu} S B X \rho] u = X_i < \frac{\rho}{\mu} B X \rho, X u > - < \frac{\rho}{\mu} B X \rho, X X_i u > =
\]

\[
= X_i \left( \frac{\rho}{\mu} b_{kj} X_j \rho X_k u + \frac{\rho}{\mu} X_i (b_{kj} X_j \rho) X_k u + \frac{\rho}{\mu} b_{kj} X_j \rho [X_i, X_k] u. \right.
\]

Now, Lemma 3.8, Lemma 3.9 and Remark 3.6 give the desired bound for the first and the second sum in the last line. To estimate the last sum we use that

\[
||[X_i, X_k] u|| \leq \frac{\alpha}{|x|} |X u|
\]

and Lemma 3.9. \( \square \)

**Lemma 2.7.** There exists a constant \( C = C(\alpha, \lambda, \Lambda, N) > 0 \) such that for \( 1 \leq i \leq N \) we have:

\[
||[X_i, -\sigma \frac{Z}{\mu}] u|| \leq C \rho |X u|.
\]
Proof. From Proposition 3.1 we have

\[ Zu = \frac{\rho}{\psi} < X\rho, Xu >. \]

Thus

\[ [X_i, \frac{\sigma}{\mu} Z]u = X_i\left(\frac{\sigma}{\psi} \frac{\rho}{\mu} < X\rho, Xu > \right) - \frac{\sigma}{\mu} \frac{\rho}{\psi} < X\rho, XX_iu > \]

\[ = X_i\left(\frac{\sigma}{\psi} \frac{\rho}{\mu} X_ku \right) + \frac{\sigma}{\mu} \frac{\rho}{\psi} X_k\rho[X_i, X_k]u \]

\[ = \frac{\rho}{\mu} X_i\left(\frac{\sigma}{\psi} \right) X_k\rho X_ku + \frac{\sigma}{\psi} \frac{\rho}{\mu} X_k\rho X_ku + \frac{\sigma}{\mu} \frac{\rho}{\psi} X_iX_k\rho X_ku + \frac{\sigma}{\mu} \frac{\rho}{\psi} X_k\rho[X_i, X_k]u. \]

Using Lemmas 3.7, 3.8, 3.5 and Proposition 3.3 together with

\[ |[X_i, X_k]u| \leq \frac{\alpha}{|x|} |Xu| \]

we can bound each of the terms above and finish the proof.

\[ \square \]

Lemma 2.8. There exists a constant \( C = C(\alpha, \lambda, \Lambda, N) > 0 \) such that

\[ |\text{div}\left(\frac{\rho}{\mu} SBX\rho\right)| \leq C\rho. \]

Proof. We have

\[ \text{div}\left(\frac{\rho}{\mu} SBX\rho\right) = < BX\rho, X(\frac{\rho}{\mu}) > + \frac{\rho}{\mu} \text{div}(SBX\rho) \]

\[ = \frac{\sigma}{\mu} - \frac{\rho}{\mu^2} ( < BX\rho, X\sigma > + < BX\rho, X\psi > ) + \frac{\rho}{\mu} X_k(b_{kj}X_j). \]

Invoking Lemmas 3.5, 3.9, Proposition 3.2 and Remark 3.6, we end the proof.

\[ \square \]

Lemma 2.9. There exists a constant \( C = C(\alpha, \lambda, \Lambda, N) > 0 \) such that

\[ |\text{div}\left(\frac{\sigma}{\mu} Z\right)| \leq C\rho. \]

Proof. The proof is straightforward after we make use of the fact that \( \psi \) is homogeneous of order 0, i.e., \( Z\psi = 0 \). Recall also that \( \text{div} Z = Q \) and \( \mu = \psi + \sigma \).
\[ \text{div} \left( \frac{\sigma}{\mu} Z \right) = Z \left( \frac{\sigma}{\mu} \right) + Q \frac{\sigma}{\mu} = Z \left( \frac{\mu - \psi}{\mu} \right) + Q \frac{\sigma}{\mu} = -Z \left( \frac{\psi}{\mu} \right) + Q \frac{\sigma}{\mu} = -\psi Z \left( \frac{1}{\mu} \right) + Q \frac{\sigma}{\mu} = \psi \frac{Z}{\mu^2} Z \sigma + Q \frac{\sigma}{\mu}. \]

Clearly
\[ |\frac{\sigma}{\mu}| \leq C \rho \psi \]
while
\[ |Z \sigma| \leq \frac{\rho}{\psi} |X \rho| X |\sigma| \leq C \rho \psi \]
by Lemma 3.5.

\[ \Box \]

**Lemma 2.10.**
\[ | < F AXu, Xu > | \leq C \rho |X u|^2. \]

**Proof.** It is enough to show that
\[ |F a_{rs}| \leq C \rho, \]
i.e.,
\[ \frac{\rho}{\psi} | < AX \rho, X a_{rs} > | \leq C \rho, \]
which is the same as
\[ | < AX \rho, X a_{rs} > | \leq C \psi \text{ for all } (r, s). \]
The assumption (H) implies
\[ |a_{ij} X_i \rho X_j a_{rs}| \leq C \psi^{\frac{1}{2}} \psi^{\frac{1}{2}} \leq C \psi, \quad n + 1 \leq j \leq N, \]
\[ |a_{ij} X_i \rho X_j a_{rs}| \leq C \left( \psi^{1 + \frac{1}{2n}} + \psi^{1 + \frac{1}{2n} \rho \psi^{\frac{1}{2}}} \right) \leq C \psi^{1 + \frac{1}{2n}} \leq C \psi, \quad 1 \leq j \leq n. \]
\[ \Box \]

We can now prove the above mentioned Rellich-type identity.

**Lemma 2.11.** Let \( X_1, \ldots, X_N \) and \( F \) be the above considered vector fields in \( \mathbb{R}^N \). We have the following identity
\[
\int_{\partial B_r} < AXu, Xu > < F, \nu > \ dH_{N-1} = \\
= 2 \int_{\partial B_r} a_{jk} X_j u < X_k, \nu > F u \ dH_{N-1} - 2 \int_{B_r} (\text{div} X_k) a_{jk} X_j u F u \ dV \\
- 2 \int_{B_r} a_{jk} X_j u [X_k, F] u \ dV + \int_{B_r} (\text{div} F) < AXu, Xu > \ dV \\
+ \int_{B_r} < (FA) Xu, Xu > \ dV - 2 \int_{B_r} F u \mathcal{L} u \ dV,
\]
where \( FA \) is the matrix with elements \( F a_{ij} \). Here, \( \nu \) denotes the outer unit normal to \( B_r \).

**Proof.** The proof of the above integral identity is based on the divergence theorem and can be carried similarly to its classical counterpart, see Ch.5 in [Ne]. Since the vector fields and the matrix \( A \) are not smooth, one has to justify the use of such result by a standard approximation argument which can be carried using the following key estimates from Lemmas 2.6 - 2.10. Specifically, Lemmas 2.8, 2.9 give

\[ |Q - \text{div} F| \leq C \rho , \]
whereas Lemmas 2.6, 2.7 imply

\[ |[X, F]u - Xu| \leq C \rho |Xu| . \]

Finally, Lemma 2.10 gives

\[ \| FA \|_{\infty} \leq C \rho . \]

\[ \square \]

**Lemma 2.12.** There exists a constant \( C_2 = C_2(\alpha, \lambda, \Lambda, N) > 0 \) such that

\[ D'(r) \geq 2 \int_{\partial B_r} \frac{1}{\mu} \frac{< AXu, Xu >^2}{|D\rho|} \ dV + \frac{Q - 2}{r} D(r) - C_2 D(r) , \]

where \( \mu \) is defined in (2.10).

**Proof.** By the co-area formula \( D(r) = \int_0^r \int_{\partial B_s} \frac{< AXu, Xu >}{|D\rho|} \ dH_{N-1} \ d s \). Hence,

\[ D'(r) = \int_{\partial B_r} \frac{< AXu, Xu >}{|D\rho|} \ dH_{N-1} = \frac{1}{r} \int_{\partial B_r} < AXu, Xu > < F, \nu > \ dH_{N-1} , \]
taking into account that on \( \partial B_r \) one has \( < F, \nu > = \frac{r}{|D\rho|} \). The latter follows from the following calculation

\[ < F, \nu > = \frac{F \rho}{|D\rho|} = \rho \frac{SAX\rho, D\rho}{|D\rho|} = \rho \frac{< AX\rho, X\rho >}{\mu |D\rho|} = \frac{r}{|D\rho|} . \]

From Lemma 2.11 we obtain
\[ D'(r) = 2 \int_{\partial B_r} \frac{1}{\mu} \frac{\langle AXu, X_\rho \rangle^2}{|D\rho|} \, dV + \frac{1}{r} \int_{B_r} (\text{div} F) \langle AXu, Xu \rangle \, dV \]

\[ - \frac{2}{r} \int_{B_r} a_{jk} X_j u [X_k, F] u \, dV + \frac{1}{r} \int_{B_r} (FA) Xu, Xu > \, dV. \]

In view of (2.13), the fact that \( \text{div} Z = Q \), and of the identities \( [X_i, Z] = X_i, i = 1, \ldots, N \), we can rewrite the above formula in the following form

\[ D'(r) = 2 \int_{\partial B_r} \frac{1}{\mu} \frac{\langle AXu, X_\rho \rangle^2}{|D\rho|} \, dH_{N-1} + \frac{Q}{r} - \frac{2}{r} D(r) \]

\[ + \frac{1}{r} \int_{B_r} \text{div}(-\frac{\sigma}{\mu} Z + \frac{\rho}{\mu} SBX \rho) \langle AXu, Xu \rangle \, dV \]

\[ - \frac{2}{r} \int_{B_r} a_{jk} X_j u [X_k, -\frac{\sigma}{\mu} Z + \frac{\rho}{\mu} SBX \rho] u \, dV \]

\[ + \frac{1}{r} \int_{B_r} (FA) Xu, Xu > \, dV. \]

At this point we are left with showing that the assumption (H) implies the correct estimates for the last three integrals. The absolute value of the integral involving the divergence is estimated by Lemmas 2.8 and 2.9. The integral involving the commutators is estimated by Lemmas 2.6 and 2.7 using eventually also the ellipticity of \( A \), cf. (1.5). Finally, the absolute value of the last integral is estimated by Lemma 2.10 and (1.5).

This finishes the proof of Lemma 2.12. \[ \square \]

3. Auxiliary results

In this section we collect some basic estimates that have been used in section two. Recall that the matrix \( S \) was defined in (2.1)

**Proposition 3.1.**

i) The following formula holds true

\[ Z = \frac{\rho}{\psi} S X \rho \]

ii) The horizontal gradient of the gauge satisfies

\[ |X_k \rho| \leq \psi^{1+\frac{1}{\alpha}} \quad \text{for} \; 1 \leq k \leq n , \]

\[ |X_{n+k} \rho| \leq (\alpha + 1) \psi^{\frac{1}{2}} \quad \text{for} \; 1 \leq k \leq m . \]
Proof. By definition
\[ X_k \rho = \psi \frac{x_k}{\rho} \quad \text{for } 1 \leq k \leq n , \]
\[ X_{n+k} \rho = (\alpha + 1)\psi \frac{y_k}{\rho^{\alpha+1}} \quad \text{for } 1 \leq k \leq m . \]
In other words, we have
\[ X_\rho = \left( \frac{\psi}{\rho} x, (\alpha + 1)\psi^{1/2} \frac{y}{\rho^{\alpha+1}} y \right) = \frac{\psi}{\rho} \left( x, (\alpha + 1)|x|^{-\alpha} y \right) = \frac{\psi}{\rho} S^{-1} Z , \]
having in mind the definition of the radial vector field \( Z \), see (2.5). From \( \frac{|x|}{\rho} = \psi^{\frac{1}{2\alpha}} \) and \( |y| \leq \rho^{\alpha+1} \) we obtain that the estimates in ii).
\[ \square \]

In the next proposition we compute the horizontal gradient of the angle function

**Proposition 3.2.** The angle function \( \psi \) satisfies the estimates
\[ |X_k \psi| \leq C \alpha \frac{\psi}{|x|} , \quad \text{if } 1 \leq k \leq n \]
\[ |X_{n+k} \psi| \leq C \alpha \frac{\psi}{\rho} , \quad \text{if } 1 \leq k \leq m . \]

Proof. Since \( \psi = \frac{|x|^{2\alpha}}{\rho^{2\alpha}} \) we have
\[ X \psi = \frac{2\alpha|x|^{2\alpha-1} X|x|}{\rho^{2\alpha}} - \frac{2\alpha|x|^{2\alpha}}{\rho^{2\alpha+1}} X \rho \]
\[ = \frac{2\alpha|x|^{2\alpha-1}}{\rho^{2\alpha}} \left( \frac{x}{|x|} 0 \right) - \frac{2\alpha|x|^{3\alpha}}{\rho^{2(2\alpha+1)}} \left( |x|^{\alpha} x \right) \]
\[ = 2\alpha \psi \left( \frac{x}{|x|^2} 0 \right) - 2\alpha \frac{\psi^2}{\rho^2} \left( (\alpha + 1)|x|^{-\alpha} y \right) . \]
This shows that
\[ X_i \psi = \begin{cases} 2\alpha \psi \frac{x_i}{|x|^2} - 2\alpha \psi^2 \frac{x_i}{\rho^2} \quad & \text{if } 1 \leq i \leq n , \\ -2\alpha(\alpha + 1)\psi \frac{y_{n+i} - x_i|x|^{\alpha}}{\rho^{\alpha+2}} \quad & \text{if } n + 1 \leq i \leq N . \end{cases} \]
Now, \( |x| \leq \rho \) and \( |y| \leq \rho^{\alpha+1} \) lead to the desired estimates.
\[ \square \]

In the proof of Theorem 1.2 the following estimates on the horizontal Hessian of \( \rho \) play an important role.
Proposition 3.3.

\[ |X_i X_j \rho| \leq C \frac{\psi}{\rho} \quad \text{for} \quad 1 \leq i, j \leq n \text{ or } n + 1 \leq i, j \leq N, \]

\[ |X_i X_{n+j} \rho| \leq C \frac{\psi^2}{|x|} = C \rho \psi^2 (\alpha + 1) \quad \text{for} \quad 1 \leq i \leq n, 1 \leq j \leq m, \]

\[ |X_{n+j} X_i \rho| \leq C \frac{\psi^2|x|}{\rho^2} = C \frac{\psi^2 + \frac{1}{\rho^2}}{\rho} \quad \text{for} \quad 1 \leq i \leq n, 1 \leq j \leq m. \]

Proof. We need to compute the second derivatives of \( \rho \) and this is done easily for example by using the product rule and the formulas from Propositions 3.1 and 3.2. We shall write only the expressions for the second derivatives.

If \( 1 \leq i, j \leq n \), we have:

\[ X_i X_j \rho = -(2\alpha + 1) \frac{\psi^2}{\rho^2} x_i x_j + 2\alpha \frac{\psi}{\rho |x|^2} x_i x_j + \frac{\psi}{\rho} \delta_{ij}. \]

If \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) we have:

\[ X_i X_{n+j} \rho = -(2\alpha + 1) \frac{\psi^2}{\rho^2} (\alpha + 1) |x|^{-\alpha} x_i y_j \]

\[ + 2\alpha \frac{\psi}{\rho |x|^2} (\alpha + 1) |x|^{-\alpha} x_i y_j - \frac{\psi}{\rho} \alpha (\alpha + 1) |x|^{-\alpha - 2} x_i y_j. \]

If \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) we have:

\[ X_{n+j} X_i \rho = -(2\alpha + 1) \frac{\psi^2}{\rho^2} (\alpha + 1) |x|^{-\alpha - 2} x_i y_j. \]

If \( 1 \leq i, j \leq m \), we have:

\[ X_{n+i} X_{n+j} \rho = -(2\alpha + 1) \frac{\psi^2}{\rho^2} (\alpha + 1) |x|^{-\alpha - 2} x_i y_j + (\alpha + 1) \frac{\psi}{\rho} \delta_{ij}. \]

At this point the estimates follow in an obvious way using \( |x| \leq \rho \) and \( |y| \leq \rho^{\alpha + 1} \). \( \square \)

Definition 3.4. Let:

\[ \mu \overset{\text{def}}{=} <AX\rho, X\rho>, \]

and also

\[ B \overset{\text{def}}{=} A - Id, \quad \sigma \overset{\text{def}}{=} <BX\rho, X\rho>. \]

One more notation we will use is: \( (b_{ij}) = B \).

Lemma 3.5. If \( (H) \) holds then:

\[ |\sigma| \leq C \rho \psi^2 \frac{1}{\rho^2}, \]

\[ |X_k \sigma| \leq C \psi^2 \quad 1 \leq k \leq N. \]
Proof. We have $\sigma = b_{ij}X_i\rho X_j\rho$. Thus Proposition 3.3 and $(H)$ give:

$$|\sigma| \leq C(\rho\psi^{1+\frac{1}{2n}} + \rho\psi^{1+\frac{1}{2n}}\psi^{\frac{1}{2}} + \rho\psi^{1+\frac{1}{2n}}\psi^{\frac{1}{2}})$$

$$\leq C(\rho\psi^{2+\frac{1}{2} + \rho}\psi^{2+\frac{1}{2} + \rho}) \leq C\rho\psi^{2+\frac{1}{2} + \rho}.$$ 

The derivatives are given by $X_k\sigma = b_{ij}X_kX_i\rho X_j\rho + X_kb_{ij}X_i\rho X_j\rho$ and we can use Propositions 3.1 and 3.3 to obtain the desired estimates.

For $1 \leq k \leq n$ we have

$$|X_k\sigma| \leq C(\rho\psi^{1+\frac{1}{2n}} + \rho\psi^{1+\frac{1}{2n}}\psi^{\frac{1}{2}})$$

$$+ \rho\psi^{1+\frac{1}{2n}}\rho\psi^{\frac{1}{2}} - \frac{1}{2n}\psi^{1+\frac{1}{2n}} + \rho\psi^{\frac{1}{2}} - \frac{1}{2n}\psi^{\frac{1}{2}})$$

$$+ C(\psi^{1+\frac{1}{2n}}\psi^{1+\frac{1}{2n}} + \psi^{\frac{1}{2}}\psi^{1+\frac{1}{2n}}\psi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}})$$

$$\leq C(\psi^{2+\frac{1}{2n}} + \psi^{2+\frac{1}{2n}} + \psi^{\frac{1}{2}}) \leq C\psi^{\frac{3}{2}}.$$ 

For $n + 1 \leq k \leq N$ we find

$$|X_k\sigma| \leq C\left(\rho\psi^{\frac{3}{2}+\frac{1}{2n}}\psi^{1+\frac{1}{2n}} + \rho\psi^{\frac{3}{2}+\frac{1}{2n}}\psi^{\frac{1}{2}}\right)$$

$$+ \rho\psi^{\frac{3}{2}+\frac{1}{2n}}\rho\psi^{\frac{1}{2}} - \frac{1}{2n}\psi^{1+\frac{1}{2n}} + \rho\psi^{\frac{1}{2}} - \frac{1}{2n}\psi^{\frac{1}{2}})$$

$$+ C\psi^{\frac{1}{2}}(\psi^{1+\frac{1}{2n}}\psi^{1+\frac{1}{2n}} + \psi^{\frac{1}{2}}\psi^{1+\frac{1}{2n}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}})$$

$$\leq C\left(\psi^{\frac{3}{2}+\frac{1}{2n}} + \psi^{\frac{3}{2}+\frac{1}{2n}} + \psi^{\frac{1}{2}}\right) \leq C\psi^{\frac{3}{2}}.$$ 

\[\square\]

Remark 3.6. Notice that a careful examination of the second part of the above proof shows that we also proved:

$$|X_kb_{ij}X_i\rho| \leq C\psi.$$
Lemma 3.7. If \((H)\) holds then:

\[
\left| X_k\left(\frac{\psi}{\mu}\right) \right| \leq C\psi^{\frac{1}{2}} \quad \text{for } 1 \leq k \leq N.
\]

Proof. It is enough to estimate the reciprocal \(\frac{\mu}{\psi}\) since

\[
X_k\left(\frac{\psi}{\mu}\right) = -\frac{\psi^2}{\mu^2} X_k\left(\frac{\mu}{\psi}\right) \quad \text{and } 0 < \lambda \leq \frac{\mu}{\psi} \leq \lambda^{-1}.
\]

From \(X_k\left(\frac{\mu}{\psi}\right) = X_k\left(\frac{\sigma}{\psi}\right)\), using Lemma 3.5 and Proposition 3.2 we obtain:

\[
\left| X_k\left(\frac{\sigma}{\psi}\right) \right| = \left| \frac{X_k\sigma}{\psi} - \frac{\sigma}{\psi^2} X_k\psi \right| \leq C\left(\psi^{\frac{1}{2}} + \frac{\rho^3}{\psi^2} \psi^{\frac{3}{2}} - \frac{1}{2} \alpha \right) = C\psi^{\frac{3}{2}}.
\]

The proof is complete. \(\square\)

Lemma 3.8. If \((H)\) holds then:

\[
\left| X_k\left(\frac{\rho}{\mu}\right) \right| \leq C\psi^{-1-\frac{1}{2n}} \quad \text{for } 1 \leq k \leq N.
\]

Proof.

\[
X_k\left(\frac{\rho}{\mu}\right) = X_k\left(\frac{\psi}{\mu} \frac{\rho}{\psi}\right) = X_k\left(\frac{\psi}{\mu}\right) \frac{\rho}{\psi} + X_k\rho - \frac{\rho X_k\psi}{\psi \mu}.
\]

Now Lemma 3.7 and Propositions 3.1 and 3.2 give:

\[
\left| X_k\left(\frac{\rho}{\mu}\right) \right| \leq C\left(\frac{\psi^{\frac{1}{2}} \rho}{\psi} + \frac{\psi^{\frac{1}{2}}}{\psi} + \frac{\rho}{\psi^2} \psi \left| x \right| \right) \leq C\psi^{-1-\frac{1}{2n}},
\]

recalling also that \(0 < \lambda \leq \frac{\mu}{\psi} \leq \lambda^{-1}\). \(\square\)

Lemma 3.9. If \((H)\) holds then:

\[
\left| b_{kj} X_j \rho \right| \leq C\rho \psi^{1+\frac{1}{2n}}
\]

Proof. If \(1 \leq j \leq n\) we have \(\left| X_j \rho \right| \leq C\psi^{1+\frac{1}{2n}}\) and \(b_{kj} \leq C\rho\). If \(n + 1 \leq j \leq N\) we have \(\left| X_j \rho \right| \leq C\psi^{\frac{3}{2}}\) and \(b_{kj} \leq C\rho \psi^{\frac{3}{2}+\frac{1}{2n}}\). \(\square\)
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