# Bound on the multiplicity of almost complete intersections[*]

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## Abstract

Let $R$ be a polynomial ring over a field of characteristic zero and let $I \subset R$ be a graded ideal of height $N$ which is minimally generated by $N+1$ homogeneous polynomials. If $I = (f_1, \ldots, f_{N+1})$ where $f_i$ has degree $d_i$ and $(f_1, \ldots, f_N)$ has height $N$, then the multiplicity of $R/I$ is bounded above by $\prod_{i=1}^{N} d_i - \max\Big\{1,\ \sum_{i=1}^{N}(d_i - 1) - (d_{N+1} - 1)\Big\}$.

*Keywords:* almost complete intersection, core, multiplicity.
*Mathematics Subject Classification:* 13B22, 13H15.

## 1 Introduction

Let $R = k[X_1, \ldots, X_n]$ where $k$ is a field of characteristic zero and let $f_1, \ldots, f_N$ be forms in $R$ of degrees $d_1, \ldots, d_N$, respectively, which generate an ideal of height $N$. That is, $f_1, \ldots, f_N$ form a regular sequence and they generate a complete intersection ideal with multiplicity $\prod_{i=1}^{N} d_i$. Let $f_{N+1}$ be yet another form, of degree $d_{N+1}$, which is a zero-divisor on $R/(f_1, \ldots, f_N)$, that is, $(f_1, \ldots, f_N) \subsetneq (f_1, \ldots, f_N) : f_{N+1} \subsetneq R$. Herein the ideal $(f_1, \ldots, f_{N+1})$ is referred to as an *almost complete intersection.* Throughout this article, let $I = (f_1, \ldots, f_{N+1})$ and $\underline{f} = f_1, \ldots, f_N$. We will abuse the notation $\underline{f}$ to also denote the ideal generated by this sequence.

It can be shown, as one would expect, that the multiplicity of $R/I$ is strictly less than that of $R/\underline{f}$. This note is aimed at making this fact more precise and exhibiting a bound for the multiplicity of $R/I$ in terms of the degrees $d_1, \ldots, d_{N+1}$ of the minimal generators of $I$. Our approach leads quite naturally

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[*] Communications in Algebra 37 (2009), doi:10.1080/00927870802278784

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to the notion of the core of an ideal, which was first alluded to by Rees and Sally in [RS] and treated more explicitly by Huneke and Swanson in [HS], and which has been the subject of growing interest in recent years.

*The core of an ideal.* Recall that given two ideals $B \subseteq A$, $B$ is said to be a *reduction* of $A$ if $A^{t+1} = BA^t$ for some non-negative integer $t$, and consequently for all integers greater than $t$. (Equivalently, a reduction of $A$ is a subideal of $A$ with the same integral closure as $A$.) The least such integer is called the *reduction number* of $A$ with respect to $B$. A reduction $B$ of $A$ is called *minimal* if no ideal properly contained in $B$ is a reduction of $A$, that is, if it is minimal with respect to inclusion. As minimal reductions are not unique, one is led to consider the intersection of all minimal reductions: the *core* of an ideal $A$, denoted $\operatorname{core}(A)$, is defined as the intersection of all reductions (equivalently, minimal reductions) of $A$.

We first take advantage of the well-understood structure of $R/\underline{f}$ and express its multiplicity $\prod_{i=1}^{N} d_i$ as the length of $R/(\underline{f},\underline{\ell})$, where $\underline{\ell} = \ell_1, \ldots, \ell_{n-N}$ is a sequence of $n-N$ general linear forms. Our focus will then be to determine when, and by how much, the additional generator $f_{N+1}$ further reduces the length of $R/(\underline{f},\underline{\ell})$ to that of $R/(I,\underline{\ell})$. Finally, as the multiplicity of $R/I$ is no greater than the length of $R/(I,\underline{\ell})$, we obtain

$$e(R/I) \;\leqslant\; \lambda(R/(I,\underline{\ell})) \;\lneqq\; \lambda(R/(\underline{f},\underline{\ell})) \;=\; e(R/\underline{f}), \tag{1}$$

where $\lambda(\ )$ and $e(\ )$ denote length and multiplicity, respectively. More concretely, we prove the following

**Theorem 1.** *Let $R$ be a polynomial ring over a field of characteristic 0. If $I \subset R$ is an almost complete intersection minimally generated by $f_1, \ldots, f_{N+1}$ such that $f_1, \ldots, f_N$ form a regular sequence, then the multiplicity of $R/I$ is at most*

$$\prod_{i=1}^{N} d_i - \max\left\{1,\ \sum_{i=1}^{N}(d_i - 1) - (d_{N+1} - 1)\right\},$$

*where $d_i = \deg(f_i)$ for $i = 1 \ldots N$.*

Let $\bar{R} := R/\underline{f}$ and let $^-$ denote the residue class in $\bar{R}$. Our argument is based on the well-known fact (see [NR]) that in $\bar{R}$ any choice of $n-N$ general linear forms $\underline{\ell}$ generates a graded minimal reduction of the homogeneous maximal ideal $\bar{\mathfrak{m}} = (\bar{X}_1, \ldots, \bar{X}_n)$, and all graded minimal reductions of $\bar{\mathfrak{m}}$ are of this form.

Note that the strict inequality in (1) holds if and only if $f_{N+1} \notin (\underline{f},\underline{\ell})$. Thus, to establish this inequality it suffices to show that the image of $f_{N+1}$ in $\bar{R}$ is not contained in some graded minimal reduction of $\bar{\mathfrak{m}}$. To this end, we appeal to a result of Corso, Polini, and Ulrich [CPU1, Theorem 4.5] which implies that in our setting $\operatorname{core}(\bar{\mathfrak{m}})$ is in fact the intersection of the graded minimal reductions of $\bar{\mathfrak{m}}$, and we show that $\bar{f}_{N+1} \notin \operatorname{core}(\bar{\mathfrak{m}})$. To compute $\operatorname{core}(\bar{\mathfrak{m}})$ we avail ourselves of a formula that was conjectured by Corso, Polini, and Ulrich in [CPU2], and was proved independently by Huneke and Trung [HT, Theorem 3.7] and Polini and Ulrich [PU, Theorem 4.5].

## 1.1 Preliminaries

We recall that given a sequence $\underline{\ell} = \ell_1, \ldots, \ell_{n-N}$ of general linear forms, the multiplicity of $R/I$ is bounded above by the length of $R/(I, \underline{\ell})$. Indeed, as $I$ has height $N$, the elements $\underline{\ell}$ constitute a system of parameters of $R/I$. The multiplicity of $R/I$ can be obtained as the Euler characteristic of $R/I$ with respect to $\underline{\ell}$

$$\begin{aligned} \chi(\underline{\ell}, R/I) &= \sum_{i \geqslant 0} (-1)^i \, \lambda\big(H_i(\underline{\ell}, R/I)\big) \\ &= \sum_{i \geqslant 0} (-1)^i \, \lambda\big(\operatorname{Tor}_i^R(R/\underline{\ell}, R/I)\big), \end{aligned}$$

where $H_\bullet(\underline{\ell}, R/I)$ denotes the Koszul homology of $\underline{\ell}$ with coefficients in $R/I$. If we further consider the first partial Euler characteristic

$$\chi_1(\underline{\ell}, R/I) = \sum_{i \geqslant 1} (-1)^{i-1} \, \lambda\big(H_i(\underline{\ell}, R/I)\big),$$

then we have $\chi(\underline{\ell}, R/I) = \lambda(R/(I, \underline{\ell})) - \chi_1(\underline{\ell}, R/I)$ and the following non-negativity result yields $e(R/I) \leqslant \lambda(R/(I, \underline{\ell}))$, as desired.

**Theorem** (Serre [S]). *The first partial Euler characteristic* $\chi_1(\underline{\ell}, R/I)$ *is non-negative, or equivalently,* $\chi(\underline{\ell}, R/I) \leqslant \lambda(R/(I, \underline{\ell}))$.

*Socle degree of complete intersections.* Recall that in our setting $\underline{f}$ is a regular sequence $f_1, \ldots, f_N$ with $\deg(f_i) = d_i$ and $\underline{\ell} = \ell_1, \ldots, \ell_{n-N}$ is a sequence of general linear forms. Next we point out that $R/(\underline{f}, \underline{\ell})$ has a pure socle generated in degree $\sum_{i=1}^{N}(d_i - 1)$. To see this, we compute $\operatorname{Tor}_n^R(R/(\underline{f}, \underline{\ell}), k)$ twice. On the one hand, resolving $k$ via the Koszul complex on $X_1, \ldots, X_n$ and tensoring with $R/(\underline{f}, \underline{\ell})$ yields

$$\operatorname{Tor}_n^R\big(R/(\underline{f}, \underline{\ell}), k\big) \cong \tfrac{(\underline{f}, \underline{\ell}) : \mathfrak{m}}{(\underline{f}, \underline{\ell})}(-n).$$

On the other hand, resolving $R/(\underline{f}, \underline{\ell})$ via the Koszul complex on $\underline{f}, \underline{\ell}$ and tensoring with $R/\mathfrak{m}$ we have

$$\operatorname{Tor}_n^R\big(R/(\underline{f}, \underline{\ell}), k\big) \cong k\big(-(n - N + \sum_{i=1}^{N} d_i)\big).$$

Hence $\operatorname{socle}(R/(\underline{f}, \underline{\ell})) \cong k\big(-(\sum_{i=1}^{N} d_i - N)\big)$, as claimed. Also note that $\lambda(R/(\underline{f}, \underline{\ell})) = e(R/\underline{f}) = \prod_{i=1}^{N} d_i$.

**Remark 2.** For brevity set $r := \sum_{i=1}^{N}(d_i - 1)$. It follows that $\mathfrak{m}^{r+1} \subseteq (\underline{f}, \underline{\ell})$, or equivalently, $\mathfrak{m}^{r+1} \equiv \underline{\ell}\,\mathfrak{m}^r$ modulo $\underline{f}$. Thus, in $\bar{R} = R/\underline{f}$ the minimal reduction $\underline{\ell}$ of $\mathfrak{m}$ has reduction number $\leqslant r$. We also note that $\bar{\mathfrak{m}} \subset \bar{R}$ is an *equimultiple* ideal, that is, its height equals its analytic spread $n - N$.

## 2 Computation of the core

The following theorem provides a formula for the core of an equimultiple ideal in a Cohen-Macaulay local ring.

**Theorem 3** (Huneke-Trung [HT, 3.7], Polini-Ulrich [PU, 4.5])**.** *Let $S$ be a Cohen-Macaulay local ring with residue field of characteristic $0$. Let $A$ be an equimultiple ideal of $S$ and let $B \subseteq A$ be a minimal reduction of $A$ with reduction number $r$. Then* $\operatorname{core}(A) = B^{r+1} : A^r$.

Note that if $A$ is an equimultiple ideal, then its minimal reduction $B$ is generated by a regular sequence and it is easily seen that $B^{r+1} : A^r = B^{t+1} : A^t$ for all $t \geqslant r$. In [PUV, Proposition 2.1] it is further shown that forming the core of zero-dimensional ideals commutes with localization. Thus, the colon formula of Theorem 3 may be applied in our setting (see also [PUV, Theorem 2.3]) and it yields

$$\operatorname{core}(\bar{\mathfrak{m}}) = \frac{(\underline{\ell}^{r+1}, \underline{f}) : \mathfrak{m}^r}{\underline{f}}, \tag{2}$$

where $r = \sum_{i=1}^{N}(d_i - 1)$ as set previously. Using (2) and minimal free resolutions, we now describe $\operatorname{core}(\bar{\mathfrak{m}})$ more precisely with the following

**Lemma 4.** *Let $R$ be a polynomial ring over a field of characteristic zero and let $\underline{f} \subset R$ be an ideal generated by a regular sequence of $N$ forms of degrees $d_1, \ldots, d_N$. Then* $\operatorname{core}(\bar{\mathfrak{m}}) = \bar{\mathfrak{m}}^{r+1}$ *with $r = \sum_{i=1}^{N}(d_i - 1)$, where $^-$ denotes the residue class in $R/\underline{f}$.*

*Proof.* Say $R = k[X_1, \ldots, X_n]$ and $\underline{f}$ is generated by the regular sequence $f_1, \ldots, f_N$ with $\deg(f_i) = d_i$. Set $\bar{R} := R/\underline{f}$ and let $\underline{\ell} = \ell_1, \ldots, \ell_{n-N}$ be a sequence of general linear forms in $R$. By Remark 2, the image of $\underline{\ell}$ in $\bar{R}$ generates a minimal reduction of $\bar{\mathfrak{m}}$ with reduction number $\leqslant r$ and by (2), $\operatorname{core}(\bar{\mathfrak{m}}) = \frac{(\underline{\ell}^{r+1}, \underline{f}) : \mathfrak{m}^r}{\underline{f}}$. To compute this colon, we first resolve $R/(\underline{\ell}^{r+1}, \underline{f})$ to determine its socle degree. Let

$$\phi := \left.\underbrace{\begin{pmatrix} \ell_1 & \cdots & \ell_{n-N} & & & & \\ & \ell_1 & & \cdots & \ell_{n-N} & & \\ & & \ddots & & & \ddots & \\ & & & \ell_1 & \cdots & \ell_{n-N} & \\ & & & & \ell_1 & \cdots & \ell_{n-N} \end{pmatrix}}_{n-N+r \text{ columns}}\right\} r+1 \text{ rows}$$

and note that the $(r+1) \times (r+1)$ minors of $\phi$ generate the ideal $\underline{\ell}^{r+1}$. Recall that $R/\underline{\ell}^{r+1}$ is perfect of grade $n-N$ and is minimally resolved by the Eagon-Northcott complex – see [EN]:

$$\begin{aligned} \mathbb{EN}(\phi): \quad & 0 \to R^{b_{n-N}}(-(n-N+r)) \xrightarrow{d_{n-N}} \cdots \\ & \cdots \to R^{b_2}(-(r+2)) \xrightarrow{d_2} R^{b_1}(-(r+1)) \xrightarrow{\wedge^{r+1}\phi} R \to R/\underline{\ell}^{r+1} \to 0. \end{aligned}$$

On the other hand, $R/\underline{f}$ is minimally resolved by the Koszul complex:

$$\mathbb{K}(\underline{f}): \quad 0 \to R(-\sum_{i=1}^{N} d_i) \to \cdots \to \bigoplus_{i=1}^{N} R(-d_i) \to R \to R/\underline{f} \to 0.$$

We consider the tensor product of the above complexes

$$\begin{aligned} \mathbb{EN}(\phi) \otimes \mathbb{K}(\underline{f}): \quad & 0 \to R^{b_{n-N}}(-(n-N+r+\sum_{i=1}^{N} d_i)) \to \cdots \\ & \cdots \to R^{b_1}(-(r+1)) \oplus \bigoplus_{i=1}^{N} R(-d_i) \to R \to R/(\underline{\ell}^{r+1}, \underline{f}) \to 0, \end{aligned}$$

and recall that its $i$-th homology is isomorphic to $\operatorname{Tor}_i^R(R/\underline{\ell}^{r+1}, R/\underline{f})$ for $i \geqslant 0$ – see [R, Theorem 11.21]. As $\underline{\ell}$ is generated by a regular sequence and $\underline{f}$ is a regular sequence modulo $\underline{\ell}$, it is also a regular sequence modulo $\underline{\ell}^{r+1}$ and all higher $\operatorname{Tor}_i$ vanish. Thus, $\mathbb{EN}(\phi) \otimes \mathbb{K}(\underline{f})$ is in fact a free resolution of $R/(\underline{\ell}^{r+1}, \underline{f})$ of length $n$ in which the $n$-th module has a twist of $-(n+r+\sum_{i=1}^{N}(d_i-1)) = -(n+2r)$. It now follows from

$$\operatorname{Tor}_n^R(R/(\underline{\ell}^{r+1}, \underline{f}), k) \;\cong\; k^{b_{n-N}}(-(n+2r)) \;\cong\; \frac{(\underline{\ell}^{r+1}, \underline{f}) : \mathfrak{m}}{(\underline{\ell}^{r+1}, \underline{f})}(-n).$$

that $R/(\underline{\ell}^{r+1}, \underline{f})$ has socle isomorphic to $k^{b_{n-N}}(-2r)$.

To prove our claim, let $x \in R$ such that $\bar{x} \in \operatorname{core}(\bar{\mathfrak{m}})$. As $\operatorname{core}(\bar{\mathfrak{m}}) = \frac{(\underline{\ell}^{r+1}, \underline{f}) : \mathfrak{m}^r}{\underline{f}}$, we have $x\,\mathfrak{m}^r \subseteq (\underline{\ell}^{r+1}, \underline{f})$ and consequently $x\,\mathfrak{m}^{r-1}$ is contained in the socle of $R/(\underline{\ell}^{r+1}, \underline{f})$. This socle is generated in degree $2r$, as shown above. Thus, $\deg x \geqslant r+1$ and $\mathrm{core}(\bar{\mathfrak{m}}) \subseteq \bar{\mathfrak{m}}^{r+1}$. The reverse inclusion is clear, as $\mathfrak{m}^{2r+1} \subseteq (\underline{\ell}^{r+1}, \underline{f})$. $\square$

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* As before, let $\bar{}$ denote the residue class in $\bar{R} = R/\underline{f}$ and let $\mathfrak{m}$ be the homogeneous maximal ideal $(X_1, \ldots, X_n)$. By Lemma 4, $\mathrm{core}(\bar{\mathfrak{m}})$ is generated in degree $r+1$, where $r = \sum_{i=1}^{N}(d_i - 1)$. So if $\deg(f_{N+1}) = d_{N+1} \leqslant r$, then $\bar{f}_{N+1} \notin \mathrm{core}(\bar{\mathfrak{m}})$, that is, $\bar{f}_{N+1}$ is not contained in some minimal reduction of $\bar{\mathfrak{m}}$.

By [CPU1, Theorem 4.5], $\mathrm{core}(\bar{\mathfrak{m}})$ is in fact the intersection of *general* minimal reductions of $\bar{\mathfrak{m}}$. As the generators of $\bar{\mathfrak{m}}$ are all in degree one, its general minimal reductions are precisely the ideals generated by $n - N$ general linear forms. Thus, we have established that $f_{N+1} \notin (\underline{f}, \underline{\ell})$ for some choice of general linear forms $\underline{\ell} = \ell_1, \ldots, \ell_{n-N}$ whenever $d_{N+1} \leqslant r$. A priori, this implies

$$\mathrm{HF}(R/(I, \underline{\ell}),\, d_{N+1}) = \mathrm{HF}\big(R/(\underline{f}, \underline{\ell}),\, d_{N+1}\big) - 1,$$

where $\mathrm{HF}(M, \ \ )$ denotes the Hilbert function of an $R$-module $M$. But as $R/(\underline{f}, \underline{\ell})$ is Gorenstein with a 1-dimensional socle in degree $r$, we have in fact

$$\mathrm{HF}(R/(I, \underline{\ell}),\, i) \leqslant \mathrm{HF}\big(R/(\underline{f}, \underline{\ell}),\, i\big) - 1 \quad \text{for} \quad d_{N+1} \leqslant i \leqslant r$$

and consequently

$$\lambda(R/(I, \underline{\ell})) \ \leqslant\ \lambda(R/(\underline{f}, \underline{\ell})) - (r - d_{N+1} + 1).$$

As shown in Section 1.1, $e(R/I) \leqslant \lambda(R/(I, \underline{\ell}))$ and we arrive at

$$e(R/I) \ \leqslant\ \prod_{i=1}^{N} d_i - \sum_{i=1}^{N}(d_i - 1) + (d_{N+1} - 1)$$

whenever $d_{N+1} \leqslant r$.

Finally, notice that the condition $d_{N+1} \leqslant r$ is equivalent to $\sum_{i=1}^{N}(d_i - 1) - (d_{N+1} - 1) \geqslant 1$. If this condition is not satisfied, one may infer by elementary means (see [E, Lemma 8]) that $e(R/I) \leqslant e(R/\underline{f}) - 1$. This proves the inequality

$$e(R/I) \ \leqslant\ \prod_{i=1}^{N} d_i - \max\left\{1,\ \sum_{i=1}^{N}(d_i - 1) - (d_{N+1} - 1)\right\}.$$

$\square$

## Acknowledgments

I thank Craig Huneke, Claudia Polini, and Bernd Ulrich for valuable conversations.

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