Bounds on Projective Dimension

by

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ABSTRACT

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In the study of ideals in polynomial rings over a field, one can glean a great deal of the properties and the invariants of a homogeneous ideal from its minimal graded free resolution. By a classical result of David Hilbert, which dates back to the nineteenth century, these resolutions are always finite and their length is bounded above by the dimension of the ring. In this thesis we consider the question whether the projective dimension (or equivalently, the Castelnuovo-Mumford regularity) of an ideal can be bounded solely in terms of its number of generators and the degrees of those generators, but independently of (the dimension of) the ring. We give an affirmative answer in the case of ideals generated by three cubic forms by showing that the projective dimension of three cubics (in any polynomial ring over a field) is at most 36. We also settle the open question of whether three cubic forms can have projective dimension greater than 4 by constructing an example with projective dimension equal to 5.

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> "As I made my way home, I thought Jem and I would get grown but there wasn't much else left for us to learn, except possibly algebra."
>
> Harper Lee, *To Kill a Mockingbird.*

Contents

List of Symbols

$\mathrm{ann}_R(M)$	annihilator of the R-module M
$\mathrm{Ass}(M)$	set of associated prime ideals of the module M
$\beta_{i,j}(M)$	(i,j)-th graded Betti number of the module M
$e(M)$	multiplicity (degree) of the module M
$\mathsf{gin}(I)$	generic initial ideal of I
$\mathrm{ht}(I)$	height of the ideal I
$\mathrm{Im}(\varphi)$	image of the homomorphism φ
$\mathsf{in}(f)$	initial term of f
$\lambda(M)$	length of the module M
$\mathrm{lcm}(f,g)$	least common multiple of f and g
$\mathrm{pd}(M)$	projective dimension of the module M
$\mathrm{reg}(M)$	Castelnuovo-Mumford regularity of the module M
revlex	degree reverse lexicographic monomial order
$\mathrm{Syz}_i(M)$	i-th syzygy module of the module M
J^{unm}	unmixed part (hull) of the ideal J

Introduction

In this thesis we study the projective dimension of homogeneous ideals in polynomial rings over a field. Throughout, R shall denote a polynomial ring $k[X_1, \ldots, X_n]$ over a field k.

Preliminaries

Below we give a rudimentary introduction to the notions of *free resolution* and *projective dimension*. For a more detailed treatment we refer the reader to [E1], [E2], or [SCH].

Free Resolutions and Projective Dimension

A *free resolution* of an R-module M can be viewed as the process of iteratively approximating the module M by simpler objects which are more easily understood, namely by free R-modules. Recall that every module is the homomorphic image of a free module. Let F_0 be a free module mapping onto M:

$$F_0 \to M \to 0. \tag{1}$$

Unless M itself is free, the map in (1) is not injective. The kernel of this map, called a (first) *syzygy module* of M, can be viewed as the error term in the first step of our approximation. This kernel, denoted here by $\mathrm{Syz}_1(M)$, is generated by the relations (the syzygies[8]) among the generators of M. If $\mathrm{Syz}_1(M)$ happens to be a free module,

[8] *Syzygy* [from Greek συζυγια] yoke, pair, copulation, conjunction

then we have succeeded in expressing M entirely in terms of free modules via the short exact sequence

$$0 \to \mathrm{Syz}_1(M) \to F_0 \to M \to 0.$$

If on the other hand $\mathrm{Syz}_1(M)$ is not free, then we enter a second stage of our approximation by mapping a free R-module F_1 onto $\mathrm{Syz}_1(M) \subset F_0$. Hence we obtain a map $F_1 \to F_0$ whose image is $\mathrm{Syz}_1(M)$ and whose kernel is called a second syzygy module of M:

$$0 \to \mathrm{Syz}_2(M) \to F_1 \to F_0 \to M \to 0.$$

Proceeding in this manner we obtain a free resolution of M, that is, an exact sequence of the form

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0, \tag{2}$$

where each F_i is a free R-module mapping onto the i-th syzygy module $\mathrm{Syz}_i(M)$, and we terminate this process at the i-th step as soon as $\mathrm{Syz}_i(M)$ itself is a free R-module. (If $\mathrm{Syz}_i(M)$ is free over R, then we need not approximate it by yet another free module.) The minimum length of such a free resolution is called the *projective dimension* of M over R, written $\mathrm{pd}_R(M)$ or $\mathrm{pd}(M)$ for short.

In our context the module M will be a quotient R/J, where J is a homogeneous ideal in R. By choosing a homogeneous set of generators for M and its syzygies in (2) and shifting the degrees of the generators of the free modules F_i accordingly, we can ensure that the maps $F_i \to F_{i-1}$ have degree zero and we obtain what is called a *graded free resolution* of M.

By constructing a free resolution as in (2), we place the R-module M inside an exact sequence of free R-modules. From this sequence we can gather many of the properties of M and extract most of its invariants (at least when the resolution is finite). Due to advances in the field of commutative and computational algebra which have led to new algorithms and improvements of old ones, alongside growing computing power of hardware in recent decades, the notion of free resolutions has made a gradual transition from a mere theoretical tool to a concrete, computable task. It is therefore of increasing interest to understand as much as possible about the complexity of this process and to expose various characteristics of free resolutions of certain classes of modules.

Hilbert Syzygy Theorem

One question that arises immediately is whether the process of a free resolution of a module M over R always terminates, in other words, whether $\mathrm{pd}(M)$ is necessarily finite. This is a well-known and fundamental result which was established by David Hilbert in the late nineteenth century:

Theorem (Hilbert Syzygy Theorem)**.** *A finitely generated graded module M over $R = k[X_1, \ldots, X_n]$ admits a finite graded free resolution of length at most n.*

Note that the above theorem not only asserts that $\mathrm{pd}_R(M)$ is finite, but that in fact it is no greater than n, the number of variables in R.

In this thesis we are motivated by the question whether $\mathrm{pd}(M)$ can be bounded, independently of n, in terms of other data associated to M. More specifically, we give an answer to a special case of the following

Question (Stillman)**.** *Is there a bound, independent of n, on the projective dimension of ideals in R which are generated by N forms of degrees $d_1, \ldots, d_N$?*

For other formulations of this question we refer the reader to Section 1.1. Setting $N = 3$ and $d_1 = d_2 = d_3 = 3$ in the above question, we prove the following

Theorem. *If $J \subset R$ is an ideal generated by three cubic forms, then* $\mathrm{pd}(R/J) \leqslant 36$.

Synopsis

Section 3.2 is chiefly devoted to the proof of the above theorem which is fairly long and at times technical. We approach the problem initially in a more general setting in Section 3.1, where we study three-generated ideals of height two (or more generally, almost complete intersections) in regular local rings. There, guided by work of Huneke and Ulrich [HU2] and Fan [F], we relate the projective dimension of R/J to the projective dimension of an ideal linked to the unmixed part of J. Recall that the unmixed part of J, written J^{unm}, is defined as the intersection of those primary components of J which have minimal height. For more on unmixed ideals and linkage we refer the reader to Sections 2.1.2 and 2.1.3.

Theorem. *Let R be a regular local ring and let J be an N-generated ideal of R with* $\mathrm{ht}(J) = N-1$. *If $\mathbf{z}$ is a maximal regular sequence in J, then* $\mathrm{pd}(R/J) \leqslant \mathrm{pd}\left({}^{R}/_{(\mathbf{z}):J}\right)+1$, *and equality holds if and only if* $\mathrm{pd}(R/J) \geqslant N$.

The above result is a pivotal part of our strategy for bounding the projective dimension of three cubics and, as $(\mathbf{z}) : J = (\mathbf{z}) : J^{\mathrm{unm}}$, it prompts us to focus our attention on the unmixed part J^{unm} which, in terms of primary decomposition, exhibits a simpler structure than J itself. To this end, we give a sharp upper bound for the multiplicity of J^{unm} in order to reduce the problem to a finite number of cases: In Section 2.2, using results of Huneke and Trung [HT] and Polini and Ulrich [PU] to compute the core of certain ideals, we show that the multiplicity of three cubics which minimally generate an ideal of height two is at most 7.

In Section 2.3 we give a classification of all height two, unmixed ideals of multiplicity two. In particular, we classify all double structures supported on a linear subspace of codimension two. This, combined with linkage theory, allows us to give a bound for the projective dimension of R/J either by bounding the projective dimension of an ideal linked to J^{unm} and applying the above theorem, or by bounding the number of linear forms needed to express the three cubics which generate J. Along the way we show that if J^{unm} contains a quadric, then $\mathrm{pd}(R/J) \leqslant 4$. Thus, the unmixed part of three cubics with projective dimension greater than 4 must be generated in degrees $\geqslant 3$.

In Section 2.4 we recall some old results [X, SD] which classify all (irreducible) varieties of multiplicity 3 and 4, alongside recent work of Brodmann and Schenzel [BSCH] on varieties of almost minimal multiplicity, and we use these as our starting point in Section 3.3 to construct three cubics of projective dimension 5.

Chapter 1

The Main Problem

1.1 Question of Stillman

The following question was originally posed by Michael Stillman and was communicated to me by Craig Huneke:

Question 1.1 (Stillman)**.** *Let k be a field. Does there exist a bound, independent of n, for the projective dimension of an arbitrary ideal $J \subset k[X_1, \ldots, X_n]$ which is generated by N forms of given degrees $d_1, \ldots, d_N$?*

Equivalently, if R denotes a polynomial ring over k in an unknown number of variables, does the module $F_1 \cong \bigoplus_{j=1}^{N} R(-d_j)$ in an arbitrary minimal graded free resolution of the form $\cdots \to F_2 \to F_1 \to R$ determine a bound on the length of this resolution?

The above question concerns a uniform bound on the projective dimension of R/J without fixing the ring R or the ideal J, but merely the number of generators of J and their degrees. In other words, it asks whether or not the following quantity is finite:

$$\sup_n \Big\{ \operatorname{pd}_R(R/J) \ \mid \ J \subset R = k[X_1, \ldots, X_n] \text{ is an ideal generated by } N \text{ forms of degrees } d_1, \ldots, d_N \Big\}.$$

In Section 1.3 we will consider the same question for the Castelnuovo-Mumford regularity of an ideal.

Reduction to an algebraically closed field. When considering Question 1.1, we may assume without loss of generality that the field k is algebraically closed. For if $\bar{k}$ denotes the algebraic closure of k, then the map $R := k[X_1, \ldots, X_n] \longrightarrow S := \bar{k}[X_1, \ldots, X_n]$ is a flat ring homomorphism and therefore $\mathrm{pd}_R(R/J) = \mathrm{pd}_S(S/JS)$. In particular, we may assume that k is infinite, which gives us the leverage to apply "prime avoidance."

Example 1.2 (The monomial case)**.** If $J \subset R$ is an ideal generated by N monomials (of arbitrary degrees), then $\mathrm{pd}(R/J) \leqslant N$. Indeed, if $m_1, \ldots, m_N$ are monomials, then the ideal $(m_1, \ldots, m_{N-1}) : m_N$ is again a monomial ideal, generated by $\frac{\mathrm{lcm}(m_i, m_N)}{m_N}$ with $i = 1 \ldots N-1$. Consider the short exact sequence

$$0 \longrightarrow \frac{R}{(m_1, \ldots, m_{N-1}) : m_N} \xrightarrow{\cdot\, m_N} \frac{R}{(m_1, \ldots, m_{N-1})} \longrightarrow \frac{R}{(m_1, \ldots, m_N)} \longrightarrow 0.$$

By induction on N, both ${}^{R}\!/_{(m_1,\ldots,m_{N-1}):m_N}$ and ${}^{R}\!/_{(m_1,\ldots,m_{N-1})}$ have projective dimension at most $N-1$. By Lemma 2.1, $\mathrm{pd}(R/(m_1, \ldots, m_N)) \leqslant N$.

Example 1.3 (Three quadrics)**.** Question 1.1 has an affirmative answer for $N = 3$ and $d_1 = d_2 = d_3 = 2$. Specifically, it was verified by Eisenbud and Huneke that if $J \subset R$ is an ideal generated by three quadric forms, then $\mathrm{pd}(R/J) \leqslant 4$. This bound is sharp: E.g. consider the ideal $(x^2, y^2, ax + by)$ or $(ab, xy, ax + by)$ in $k[x, y, a, b]$.

Three cubics. In contrast to Example 1.3, it was not known until now whether ideals generated by three cubic forms necessarily have bounded projective dimension. Perhaps more intriguingly, there were no known examples of three cubics f, g, h with $\mathrm{pd}(R/(f, g, h))$ greater than 4. In Section 3.2 we settle the former question by proving the existence of a uniform bound of 36, and in Section 3.3 we utilize our methods to derive an example of three cubics f, g, h with $\mathrm{pd}(R/(f, g, h)) = 5$.

1.2 The Construction of Burch

Given that the ring $R = k[X_1, \ldots, X_n]$ in Question 1.1 is not fixed (in other words, n can be arbitrarily large), it goes without saying that in order to bound the projective dimension it is necessary to restrict the number of generators of the ideals considered. It is perhaps not entirely obvious, however, that merely restricting the number of generators is far from sufficient. In this section we recall a classical result of Burch and Kohn on three-generated ideals which effectively demonstrates this fact. We also describe Burch's construction of three-generated ideals with prescribed, arbitrarily large projective dimension and point out how this construction affects the degrees of the generators.

The following theorem, stated here in its generality, is due to Burch in the local case and was generalized by Kohn to the global case. In our context, it states that any integer can be attained as the projective dimension of a three-generated ideal, and consequently as that of an N-generated ideal.

Theorem 1.4 (Burch [B], Kohn [K]). *Let S be a commutative Noetherian ring and let M be a finitely generated S-module, possibly of infinite projective dimension. There exists a three-generated ideal $A \subset S$ such that* $\operatorname{pd}(S/A) = \operatorname{pd}(M)$.

We outline Burch's proof of Theorem 1.4 by constructing a three-generated ideal of given projective dimension s. Set $R = k[X_1, \ldots, X_{2s-4}]$ with $s \geqslant 4$ and consider the ideal

$$I = (X_1, X_2) \cap (X_3, X_4) \cap \cdots \cap (X_{2s-5}, X_{2s-4}). \tag{1.1}$$

Observe that $\operatorname{pd}(R/I) = s - 1$. Indeed, if we let $I' = (X_1, X_2) \cap \cdots \cap (X_{2s-7}, X_{2s-6})$, then $I = I' \cap (X_{2s-5}, X_{2s-4})$ and we may assume that $\operatorname{pd}(R/I') = s - 2$ by induction on s. And as the indeterminates X_{2s-5}, X_{2s-4} form a regular sequence on R/I', we have $\operatorname{pd}\big({}^{R}/_{I'+(X_{2s-5}, X_{2s-4})}\big) = s$. Applying Lemma 2.1 to the short exact sequence

$$0 \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I'} \oplus \frac{R}{(X_{2s-5}, X_{2s-4})} \longrightarrow \frac{R}{I' + (X_{2s-5}, X_{2s-4})} \longrightarrow 0$$

yields $\operatorname{pd}(R/I) = s - 1$, as claimed.

Now set $f_1 := X_1X_3 \cdots X_{2s-5}$ and $f_2 := X_2X_4 \cdots X_{2s-4}$ and note that the ideal (f_1, f_2) has $(s-2)^2$ associated prime ideals of the form (X_i, X_j) with odd i and even j. By primary decomposition, $(f_1, f_2) = I \cap Q$, where $Q \not\subseteq (X_{2k-1}, X_{2k})$ for $k = 1 \dots s-2$ and by prime avoidance, $Q \not\subseteq \bigcup_{k=1}^{s-2}(X_{2k-1}, X_{2k})$. Choose an element $f_3 \in Q$ such that $f_3 \notin \bigcup_{k=1}^{s-2}(X_{2k-1}, X_{2k})$, that is, f_3 avoids all prime ideals associated to I. So $f_3 \in Q$ is a non-zerodivisor modulo I and we have

$$\begin{aligned}(f_1, f_2) : f_3 &= (I \cap Q) : f_3 \\ &= I : f_3 \\ &= I.\end{aligned} \tag{1.2}$$

The above equality leads us to the short exact sequence

$$0 \longrightarrow \frac{R}{I} \xrightarrow{\cdot\, f_3} \frac{R}{(f_1, f_2)} \longrightarrow \frac{R}{(f_1, f_2, f_3)} \longrightarrow 0.$$

We have shown that $\mathrm{pd}(R/I) = s - 1$ and as f_1, f_2 do not have a common divisor, they form a regular sequence on R and $\mathrm{pd}(R/(f_1, f_2)) = 2$. Applying Lemma 2.1 to the above short exact sequence now yields $\mathrm{pd}(R/(f_1, f_2, f_3)) = s$, which concludes the argument.

1.2.1 The Unmixed Part of Burch's Ideal

One cautionary note about the primary decomposition of (f_1, f_2, f_3): As f_3 was chosen from an intersection of minimal primes of (f_1, f_2), we have $\mathrm{ht}(f_1, f_2, f_3) = 2$. And clearly, $(f_1, f_2) : f_3 = (f_1, f_2) : (f_1, f_2, f_3)$. So, given the equality (1.2), one might be misled into thinking that the ideal I is linked[1] to (f_1, f_2, f_3). We note that this is not the case, since the ideal (f_1, f_2, f_3) is not unmixed[1]. However, as shown later on in Lemma 2.3, one always has

$$(f_1, f_2) : (f_1, f_2, f_3) = (f_1, f_2) : (f_1, f_2, f_3)^{\mathrm{unm}},$$

where $(\quad)^{\mathrm{unm}}$ denotes the unmixed part of an ideal. That is, the ideal I is linked to the unmixed part of (f_1, f_2, f_3). In particular, by linking back, we can compute this

[1]See Sections 2.1.2 and 2.1.3 for the definitions of unmixedness and linkage.

unmixed part which turns out to be the ideal Q — see Remark 2.5.

$$\begin{aligned}(f_1, f_2, f_3)^{\mathrm{unm}} &= (f_1, f_2) : \big((f_1, f_2) : f_3\big) \\ &= (f_1, f_2) : I \\ &= (I \cap Q) : I \\ &= Q.\end{aligned}$$

Next we examine the ideal Q more closely.

1.2.2 The Degrees of the Generators

One crucial aspect of Burch's construction is that it imposes lower bounds on the degrees of the generators f_1, f_2, f_3. It is evident that the first two generators f_1 and f_2 have degree $s-2$. As for the third generator f_3, it is worth noting that the ideal Q to which it belongs is the intersection of $(s-2)(s-3)$ prime ideals of the form (X_i, X_j), namely those with odd index i and even index j which do not occur in the decomposition (1.1) of I. To see how this affects the degree of f_3, we give an explicit description of the ideal Q in terms of its generators.

Set $g_k := X_1 X_2 \cdots \hat{X}_{2k-1} \hat{X}_{2k} \cdots X_{2s-5} X_{2s-4}$ with $k = 1 \ldots s-2$. (Here $\hat{}$ indicates the omission of a term.) We claim that $Q = (g_1, \ldots, g_{s-2}) + (f_1, f_2)$. Writing

$$Q = Q' \cap \bigcap_{i=1}^{s-3} (X_{2i-1}, X_{2s-4}) \cap \bigcap_{j=1}^{s-3} (X_{2s-5}, X_{2j}),$$

we may assume by induction on s that $Q' = (g'_1, \ldots, g'_{s-3}) + (f'_1, f'_2)$, where

$$\begin{aligned} f'_1 &= X_1 X_3 \cdots X_{2s-7}, \quad f'_2 = X_2 X_4 \cdots X_{2s-6}, \\ g'_k &= X_1 X_2 \cdots \hat{X}_{2k-1} \hat{X}_{2k} \cdots X_{2s-7} X_{2s-6} \end{aligned}$$

with $k = 1 \ldots s-3$. It is easily seen that

$$\bigcap_{i=1}^{s-3} (X_{2i-1}, X_{2s-4}) = (f'_1, X_{2s-4}) \quad \text{and} \quad \bigcap_{j=1}^{s-3} (X_{2s-5}, X_{2j}) = (X_{2s-5}, f'_2).$$

Further, the intersection $(f'_1, X_{2s-4}) \cap (f'_2, X_{2s-5})$ equals $(g_{s-2}, f_1, f_2, X_{2s-5} X_{2s-4})$. This and the fact that the elements g_{s-2}, f_1, f_2 are all divisible by either f'_1 or f'_2

brings us to

$$\begin{aligned} Q &= (f_1', f_2', g_1', \ldots, g_{s-3}') \cap (g_{s-2}, f_1, f_2, X_{2s-5}X_{2s-4}) \\ &= (f_1', f_2', g_1', \ldots, g_{s-3}') \cap (X_{2s-5}X_{2s-4}) + (g_{s-2}, f_1, f_2) \\ &= (f_1X_{2s-4}, f_2X_{2s-5}, g_1, \ldots, g_{s-3}) + (g_{s-2}, f_1, f_2) \\ &= (g_1, \ldots, g_{s-2}, f_1, f_2). \end{aligned}$$

So, modulo (f_1, f_2), the ideal Q is generated by $s-2$ forms, each of degree $2s-6$. This is the least possible degree any choice of f_3 can have in Burch's construction. In fact, any candidate for f_3 must have a non-zero contribution from each g_k in order to ensure that the condition $f_3 \notin \bigcup_{k=1}^{s-2}(X_{2k-1}, X_{2k})$ is satisfied. Thus, the simplest choice for f_3, in terms of complexity, would be $g_1 + \cdots + g_{s-2}$.

Example 1.5. We construct a three-generated ideal of projective dimension 5. Let R be the polynomial ring $k[a,b,c,d,e,f]$ and choose the cubics ace, bdf as the first two generators. Then

$$(ace, bdf) = (a,b) \cap (c,d) \cap (e,f) \cap (ace, bdf, abcd, abef, cdef).$$

Choosing the quartic $abcd + abef + cdef$ as the third generator, we get

$$\operatorname{pd}(R/(ace,\, bdf,\, abcd + abef + cdef)) = 5.$$

1.3 Bound on Castelnuovo-Mumford Regularity

While the projective dimension of a graded R-module M is the minimum length of a resolution of M by free R-modules and it specifies in how many steps M can be resolved, the *Castelnuovo-Mumford regularity* of M can be viewed as the width of a minimal graded free resolution of M over R and it gives an indication of the complexity of the process. This analogy gains justification in view of the so-called *Betti diagram* of M — see for example [E2]:

	0	1	$\cdots\cdots$	i
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots\cdots$	$\beta_{i,i}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots\cdots$	$\beta_{i,i+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
j	$\beta_{0,j}$	$\beta_{1,j+1}$	$\cdots\cdots$	$\beta_{i,i+j}$

Here $\beta_{i,j} = \beta_{i,j}(M) := \dim_k \operatorname{Tor}_i^R(M,k)_j$ are the graded *Betti numbers* of M and the regularity of M is defined as $\operatorname{reg}(M) := \max\{\ j-i \mid \beta_{i,j}(M) \neq 0 \text{ for some } i\ \}$. Note that the Betti number $\beta_{i,j}$ is plotted in the i-th column, but the $(j-i)$-th row of the Betti diagram. This pattern, as well as the definition of regularity, takes into account the fact that in a minimal graded free resolution $\beta_{i,j} = 0$ whenever $i > j$.

The Castelnuovo-Mumford regularity is the homological extension of the notion of 'degree' and, unlike the maximal degree of a minimal generator, it behaves well on short exact sequences in a manner entirely similar to that of projective dimension — simply replace -1 with $+1$ and vice versa in the statement of Lemma 2.1. It follows from the definition that $d(M) \leqslant \operatorname{reg}(M)$ where $d(M)$ denotes the maximal degree of a minimal generator of M, and that $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$ for an ideal $I \subset R$.

In analogy to Question 1.1, it would seem reasonable to raise the following

Question 1.6. *Is there a bound on the regularity of an ideal solely in terms of the number of its generators and the degrees of those generators?*

As we shall outline below, this question is indeed equivalent to Question 1.1. The following argument is due to Giulio Caviglia.

Suppose Question 1.1 has an affirmative answer, that is, there is a bound $B = B(N; d_1, \dots, d_N)$ such that $\mathrm{pd}(R/J) \leqslant B$ for any ideal $J \subset R = k[X_1, \dots, X_n]$ which is (minimally) generated by N forms of degrees $d_1, \dots, d_N$. By the Auslander-Buchsbaum formula we have $\mathrm{depth}(R/J) \geqslant n - B$. Let $\underline{\ell} = l_1, \dots, l_{n-B}$ be $n - B$ general linear forms and consider the quotient $R/\underline{\ell}$ which is a polynomial ring in B variables. Note that going modulo $\underline{\ell}$ leaves the regularity of R/J unchanged. Indeed, if $x \in R$ is a homogeneous element which is a non-zerodivisor modulo an ideal I, that is, $I : x = I$, then the short exact sequence

$$0 \longrightarrow \frac{R}{I}\big(-\deg(x)\big) \xrightarrow{\ \cdot\, x\ } \frac{R}{I} \longrightarrow \frac{R}{I + (x)} \longrightarrow 0$$

yields $\mathrm{reg}\big({}^{R}\!/_{I+(x)}\big) = \mathrm{reg}(R/I) + \deg(x) - 1$. So, by induction, $\mathrm{reg}(R/J) = \mathrm{reg}\big({}^{R}\!/_{J+(\underline{\ell})}\big)$. But now, over $R/\underline{\ell}$, there exists a (doubly exponential) bound for the regularity of ${}^{R}\!/_{J+(\underline{\ell})}$ in terms of $d(J) = \max\{d_1, \dots, d_N\}$ and the number of variables B — see for example [BM, Theorem 3.7]. This bounds $\mathrm{reg}(R/J)$ in terms of the original data N and $d_1, \dots, d_N$.

Conversely, assume that Question 1.6 can be answered in the positive, that is, there exists a bound $B = B(N; d_1, \dots, d_N)$ such that $\mathrm{reg}(J) \leqslant B$ for any ideal $J \subset R$ which is generated by N forms of degrees $d_1, \dots, d_N$. Consider $\mathsf{gin}_{\mathrm{revlex}}(J)$, the generic initial ideal of J with respect to the degree reverse lexicographic monomial order (revlex, for short). The choice of this particular monomial order is crucial: By a theorem of Bayer and Stillman [E1, Corollaries 19.11 and 20.21], passing to the generic initial ideal under revlex preserves the projective dimension as well as the regularity of an ideal:

$$\mathrm{pd}(R/J) = \mathrm{pd}(R/\mathsf{gin}_{\mathrm{revlex}}(J)) \quad \text{and} \quad \mathrm{reg}(R/J) = \mathrm{reg}(R/\mathsf{gin}_{\mathrm{revlex}}(J)).$$

(See [BCP] for generalizations of these results.) Moreover, the projective dimension can be read off directly as the largest s for which the variable X_s divides some minimal generator of $\mathsf{gin}_{\mathrm{revlex}}(J)$, or equivalently, as the number of distinct variables appearing in all the monomials minimally generating $\mathsf{gin}_{\mathrm{revlex}}(J)$. (Recall that every monomial ideal has a unique minimal set of monomial generators.) This characterization of projective dimension allows us to bound $\mathrm{pd}(R/J)$ in terms of the data $(N; d_1, \dots, d_N)$

and the number of generators of $\mathsf{gin}_{\mathrm{revlex}}(J)$, as follows:

$$\begin{aligned}
\mathrm{pd}(R/J) &= \mathrm{pd}(R/\mathsf{gin}_{\mathrm{revlex}}(J)) \\
&= \text{number of variables appearing in generators of } \mathsf{gin}_{\mathrm{revlex}}(J) \\
&\leqslant \text{sum of the degrees of generators of } \mathsf{gin}_{\mathrm{revlex}}(J) \\
&\leqslant \Big(\text{number of generators of } \mathsf{gin}_{\mathrm{revlex}}(J)\Big) \cdot d(\mathsf{gin}_{\mathrm{revlex}}(J)) \\
&\leqslant \Big(\text{number of generators of } \mathsf{gin}_{\mathrm{revlex}}(J)\Big) \cdot \mathrm{reg}(\mathsf{gin}_{\mathrm{revlex}}(J)) \\
&= \Big(\text{number of generators of } \mathsf{gin}_{\mathrm{revlex}}(J)\Big) \cdot \mathrm{reg}(J) \\
&\leqslant \Big(\text{number of generators of } \mathsf{gin}_{\mathrm{revlex}}(J)\Big) \cdot B(N; d_1, \ldots, d_N).
\end{aligned}$$

(In characteristic zero, the regularity of a generic initial ideal (with respect to any monomial order) is equal to the maximal degree of a minimal generator — see [BS, Proposition 2.9]. That is, the third inequality above becomes an equality. However, we shall not make any assumptions on the characteristic of the field k in this section.) Next we bound the number of generators of $\mathsf{gin}_{\mathrm{revlex}}(J)$ in terms of $(N; d_1, \ldots, d_N)$.

By definition of a Gröbner basis, the ideal $\mathsf{gin}_{\mathrm{revlex}}(J)$ is generated by the initial terms of the elements of a Gröbner basis of J, after a generic change of coordinates. Note that a change of coordinates does not affect the invariants of an ideal. So, without loss of generality, we assume that J is in generic coordinates. To bound the number of generators of $\mathsf{gin}_{\mathrm{revlex}}(J)$, we show that the cardinality of a Gröbner basis of J, using revlex, is bounded in terms of $(N; d_1, \ldots, d_N)$.

Recall that a Gröbner basis is constructed by the Buchberger Algorithm from a generating set through the process of adjoining S-pairs of the form

$$\mathrm{S}(f, g) := \frac{\mathrm{lcm}(\mathsf{in}(f), \mathsf{in}(g))}{\mathsf{in}(f)}\, f - \frac{\mathrm{lcm}(\mathsf{in}(f), \mathsf{in}(g))}{\mathsf{in}(g)}\, g.$$

Now, starting with a generating set of N elements, the maximum number of possible S-pairs adjoined in every iteration of the Buchberger algorithm is a (polynomial) function of N. E.g. in the first iteration there are at most $i_1 = \binom{N}{2}$ new S-pairs, in the second iteration there are at most $\binom{N+i_1}{2} - i_1 = \binom{i_1}{2} + i_1 N$ new S-pairs, and so on.

As for the number of possible iterations, note that on the one hand we have $\deg\big(\mathrm{S}(f,g)\big) \geqslant \max\{\deg(f), \deg(g)\}$ and this inequality is strict unless $\mathsf{in}(f)$ divides $\mathsf{in}(g)$ or vice versa. On the other hand,

$$\begin{aligned}\deg\big(\mathrm{S}(f,g)\big) &= \deg\big(\mathsf{in}(\mathrm{S}(f,g))\big)\\ &\leqslant d\big(\mathsf{gin}_{\mathrm{revlex}}(J)\big)\\ &\leqslant \mathrm{reg}\big(\mathsf{gin}_{\mathrm{revlex}}(J)\big)\\ &= \mathrm{reg}(J)\\ &\leqslant B.\end{aligned}$$

That is, the degrees of the new S-pairs increase with every iteration of the Buchberger algorithm, while at the same time they cannot surpass $B = B(N; d_1, \ldots, d_N)$. This limits the number of possible iterations in terms of $(N; d_1, \ldots, d_N)$.

So, in generic coordinates, the number of elements in a Gröbner basis of J using revlex can be bounded in terms of $(N; d_1, \ldots, d_N)$ and hence the same holds for the number of generators of $\mathsf{gin}_{\mathrm{revlex}}(J)$. This fact, along with the preceding estimate on $\mathrm{pd}(R/J)$, implies the existence of a bound for $\mathrm{pd}(R/J)$ in terms of the original data N and $d_1, \ldots, d_N$.

Chapter 2

Basic Tools

2.1 Standard Facts

2.1.1 Projective Dimension on Short Exact Sequences

The following well-known fact, equivalent to the so-called *depth lemma*, describes how projective dimension behaves on short exact sequences. Completely analogous statements hold for other homological invariants such as grade or Castelnuovo-Mumford regularity.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a local or positively graded ring. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finite R-modules. The following inequalities hold:*

$$\begin{aligned} \mathrm{pd}(A) &\leqslant \max\{\mathrm{pd}(B)\,,\, \mathrm{pd}(C) - 1\}, \\ \mathrm{pd}(B) &\leqslant \max\{\mathrm{pd}(A)\,,\, \mathrm{pd}(C)\}, \\ \mathrm{pd}(C) &\leqslant \max\{\mathrm{pd}(A) + 1,\, \mathrm{pd}(B)\}. \end{aligned}$$

Proof. The above inequalities follow readily from the long exact sequence on $\mathrm{Tor}_\bullet^R(k, -)$ induced by the short exact sequence $0 \to A \to B \to C \to 0$, that is,

$$\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(k, C) \longrightarrow \mathrm{Tor}_i^R(k, A) \longrightarrow \mathrm{Tor}_i^R(k, B) \longrightarrow \cdots$$

where $k = R/\mathfrak{m}$. □

In particular, if any two modules in a short exact sequence have finite projective dimension, then so does the third module.

In certain frequently occurring instances, it is easily seen that the inequalities of Lemma 2.1 are in fact equalities:

$$\begin{aligned} \mathrm{pd}(B) \leqslant \mathrm{pd}(C) - 1 \quad &\Longrightarrow \quad \mathrm{pd}(A) = \mathrm{pd}(C) - 1, \\ \mathrm{pd}(C) \leqslant \mathrm{pd}(A) \quad &\Longrightarrow \quad \mathrm{pd}(B) = \mathrm{pd}(A)\,, \\ \mathrm{pd}(A) + 1 \leqslant \mathrm{pd}(B) \quad &\Longrightarrow \quad \mathrm{pd}(C) = \mathrm{pd}(B)\,. \end{aligned} \tag{2.1}$$

The proofs of the above implications are straight forward: Suppose $\mathrm{pd}(B) \leqslant \mathrm{pd}(C)-1$. Then $\mathrm{pd}(A) \leqslant \mathrm{pd}(C) - 1$ by the first inequality in Lemma 2.1 and, as $\mathrm{pd}(C) \nleqslant \mathrm{pd}(B)$, we must have $\mathrm{pd}(C) \leqslant \mathrm{pd}(A) + 1$ by the third inequality. Thus, $\mathrm{pd}(A) = \mathrm{pd}(C) - 1$. The other two implications are proved similarly.

It is perhaps worth noting that one of the following identities is always satisfied:

$$\begin{aligned} \text{either} \quad & \mathrm{pd}(A) = \mathrm{pd}(B)\,, \\ \text{or} \quad & \mathrm{pd}(B) = \mathrm{pd}(C)\,, \\ \text{or} \quad & \mathrm{pd}(C) = \mathrm{pd}(A) + 1. \end{aligned}$$

To verify this directly, suppose $\mathrm{pd}(A) \neq \mathrm{pd}(B)$. If $\mathrm{pd}(A) < \mathrm{pd}(B)$, then the second inequality in Lemma 2.1 implies $\mathrm{pd}(B) \leqslant \mathrm{pd}(C)$, while the third inequality yields $\mathrm{pd}(C) \leqslant \mathrm{pd}(B)$, hence $\mathrm{pd}(B) = \mathrm{pd}(C)$. If on the other hand $\mathrm{pd}(A) > \mathrm{pd}(B)$, then the first inequality in Lemma 2.1 implies $\mathrm{pd}(A) \leqslant \mathrm{pd}(C) - 1$, while the third inequality yields $\mathrm{pd}(C) \leqslant \mathrm{pd}(A) + 1$, hence $\mathrm{pd}(C) = \mathrm{pd}(A) + 1$.

2.1.2 Unmixed Ideals

Definition. The *unmixed part* or *unmixed hull* of an ideal J, denoted by J^{unm}, is the intersection of its primary components of minimal height (or maximal dimension). An ideal is said to be *unmixed* if it equals its unmixed part. By primary decomposition, J^{unm} has the same height as J and

$$J \;=\; J^{\mathrm{unm}} \;\cap\; \{\text{ primary components of height} > \mathrm{ht}(J)\ \}.$$

In particular, $J_P = J_P^{\mathrm{unm}}$ for any prime ideal P such that $\mathrm{ht}(P) \leqslant \mathrm{ht}(J)$. Note that in general the unmixed part is larger than the intersection of the minimal primary components: E.g. the unmixed part of the radical ideal $(XY, XZ) = (X) \cap (Y, Z)$ in $k[X, Y, Z]$ is (X).

On occasion we will employ the following

Lemma 2.2. *If $J \subseteq I$ are two ideals with $\mathrm{ht}(J) = \mathrm{ht}(I)$, then $J^{\mathrm{unm}} \subseteq I^{\mathrm{unm}}$.*

(The assumption on height is necessary; e.g. consider the ideals $J = (X) \cap (Y, Z)$ with $J^{\mathrm{unm}} = (X)$ and $I = I^{\mathrm{unm}} = (Y, Z)$.)

Proof. The statement of the lemma is a consequence of the following more general fact: Every primary component of I of minimal height contains the corresponding primary component of J. More precisely, if $P \in \mathrm{Ass}(R/I)$ with $\mathrm{ht}(P) = \mathrm{ht}(I)$, then clearly $P \in \mathrm{Ass}(R/J)$ with $\mathrm{ht}(P) = \mathrm{ht}(J)$ by our hypotheses. So if $I^{\mathrm{unm}} = K_1 \cap \cdots \cap K_m$ with $\sqrt{K_i} = P_i$, then $J^{\mathrm{unm}} = L_1 \cap \cdots \cap L_m \cap L_{m+1} \cap \cdots \cap L_n$ with $\sqrt{L_i} = P_i$ for $i = 1 \ldots m$. Now, showing $L_i \subseteq K_i$ for $i = 1 \ldots m$ is sufficient to prove the inclusion $J^{\mathrm{unm}} \subseteq I^{\mathrm{unm}}$.

Since P_i is a minimal prime of J and I, localizing $J \subseteq I$ at P_i yields $(L_i)_{P_i} \subseteq (K_i)_{P_i}$. As L_i and K_i are primary to P_i, this entails $L_i \subseteq K_i$. □

We also make the following observation for future reference. The proof makes use of what is often referred to as the *local-global principle*: If $J \subseteq I$ are two ideals such that $J_P = I_P$ locally at every $P \in \mathrm{Ass}(R/J)$, then $J = I$ globally.

Lemma 2.3. *Let K be unmixed and let J be an ideal with $\mathrm{ht}(J) \geqslant \mathrm{ht}(K)$. Then $K : J^{\mathrm{unm}} = K : J$.*

Proof. As $J \subseteq J^{\mathrm{unm}}$, we have $K : J^{\mathrm{unm}} \subseteq K : J$. So it suffices to check the claim locally at the associated primes P of the ideal $K : J^{\mathrm{unm}}$. But $\mathrm{Ass}(R/(K : J^{\mathrm{unm}})) \subseteq \mathrm{Ass}(R/K)$. So $\mathrm{ht}(P) \leqslant \mathrm{ht}(J)$ by our assumption and the claim follows simply from $J_P^{\mathrm{unm}} = J_P$. □

2.1.3 Linkage

Recall the notion of *algebraic linkage* as introduced by Peskine and Szpiro in [PS]:

Definition. Two proper ideals A and B of height g in a Cohen-Macaulay ring S are said to be *(directly) linked* if there exists a maximal regular sequence $\mathbf{z} = z_1, \ldots, z_g$ in $A \cap B$ such that $A = (\mathbf{z}) : B$ and $B = (\mathbf{z}) : A$.

The ideals A and B in the above definition are necessarily unmixed of height g, since this is the case for the ideal $(\mathbf{z})$ by Macaulay's Unmixedness Theorem and $\operatorname{Ass}(S/(\mathbf{z})) = \operatorname{Ass}(S/A) \cup \operatorname{Ass}(S/B)$. The multiplicities of these ideals are complementary to each other in the sense that $e(S/(\mathbf{z})) = e(S/A) + e(S/B)$.

If the underlying ring is Gorenstein, then the unmixedness property is also sufficient for an ideal to be linked to some other ideal. More precisely, the following fundamental result of linkage theory asserts that in a Gorenstein ring one can always produce a link to an unmixed ideal and that the Cohen-Macaulay property is preserved by this process.

Proposition 2.4 (Peskine-Szpiro [PS], [HU1, Proposition 2.5])**.** *Let A be an unmixed ideal of height g in a (not necessarily local) Cohen-Macaulay ring S such that S_P is Gorenstein for all $P \in \operatorname{Ass}(S/A)$. Let $\mathbf{z} = z_1, \ldots, z_g$ be a maximal regular sequence inside A with $(\mathbf{z}) \neq A$ and set $B = (\mathbf{z}) : A$. Then*

(a) $A = (\mathbf{z}) : B$, that is, A and B are linked.

(b) S/A is Cohen-Macaulay if and only if S/B is Cohen-Macaulay.

(c) If S is local and S/A is Cohen-Macaulay, then S/A has finite projective dimension if and only if S/B has finite projective dimension.

Remark 2.5. As an immediate consequence of Lemma 2.3 and part (a) of Proposition 2.4 we observe that if $\mathbf{z}$ is any maximal regular sequence inside an ideal J in a Gorenstein ring, then $(\mathbf{z}) : J$ is linked to the unmixed part of J. In particular, for any choice of $\mathbf{z}$, we can compute the unmixed part of J by linking back:

$$J^{\text{unm}} = (\mathbf{z}) : \big((\mathbf{z}) : J\big). \tag{2.2}$$

Given the natural isomorphisms ${}^{(\mathbf{z}):J}/_{(\mathbf{z})} \cong \operatorname{Hom}_R(R/J, R/(\mathbf{z})) \cong \operatorname{Ext}^g_R(R/J, R)$, with $g = \operatorname{ht}(J)$, the formula (2.2) agrees with the result in [EHV, Theorem 1.1] which states that if R is a regular ring, then $J^{\text{unm}} = \operatorname{ann}_R \operatorname{Ext}^g_R(R/J, R)$.

While two linked ideals in general do not have the same projective dimension, we shall show below that any two links of an ideal in a Gorenstein ring do necessarily have the same projective dimension. To this end, we recall the notion of the *even linkage class* of an unmixed ideal I as the set of all ideals which can be obtained from I by an even number of links. In particular, by part (a) of Proposition 2.4, I is a member of its own even linkage class as it is evenly linked to itself.

Lemma 2.6. *Let R be a Gorenstein local ring and let $I \subset R$ be an unmixed ideal. All ideals which are linked to I have the same (finite or infinite) projective dimension. Equivalently, all ideals in the even linkage class of an unmixed ideal have the same projective dimension.*

Proof. Let I be an unmixed ideal of R and let I' be an ideal linked to I. By definition, I' is of the form $(\mathbf{z}) : I$ with some maximal regular sequence $\mathbf{z} = z_1, \ldots, z_g$ in I, where $g = \operatorname{grade}(I) = \operatorname{grade}(I')$. We need to show that $\operatorname{pd}(R/I')$ is independent of the particular choice of $\mathbf{z}$.

First suppose R/I is Cohen-Macaulay. In this case it follows from Proposition 2.4 that all links of I have the same (finite or infinite) projective dimension as I. In other words, in the Cohen-Macaulay case $\operatorname{pd}(-)$ is constant on the entire linkage class of I, rather than just on the even linkage class. Indeed, if R/I is Cohen-Macaulay, then so is R/I' by part (b) of Proposition 2.4 and $\operatorname{pd}(R/I)$ is finite if and only if $\operatorname{pd}(R/I')$ is finite by part (c), in which case $\operatorname{pd}(R/I)$ and $\operatorname{pd}(R/I')$ both equal g by the Auslander-Buchsbaum formula.

If on the other hand R/I is not Cohen-Macaulay, then neither is R/I' by part (b) of Proposition 2.4. In particular, $\operatorname{pd}(R/I')$ is strictly greater than g, possibly infinite. Consider the short exact sequence

$$0 \longrightarrow \frac{(\mathbf{z}) : I}{(\mathbf{z})} \longrightarrow \frac{R}{(\mathbf{z})} \longrightarrow \frac{R}{I'} \longrightarrow 0$$

and note that the middle term $R/(\mathbf{z})$ is resolved by the Koszul complex on $z_1, \ldots, z_g$ and $\operatorname{pd}(R/(\mathbf{z})) = g$. So $\operatorname{pd}(R/(\mathbf{z})) \leqslant \operatorname{pd}(R/I') - 1$ and therefore $\operatorname{pd}\big({}^{(\mathbf{z}):I}/_{(\mathbf{z})}\big) = \operatorname{pd}(R/I') - 1$ by Lemma 2.1 — see (2.1) on page 16. (Here $\operatorname{pd}\big({}^{(\mathbf{z}):I}/_{(\mathbf{z})}\big)$ is finite if and only if $\operatorname{pd}(R/I')$ is finite.) But ${}^{(\mathbf{z}):I}/_{(\mathbf{z})}$ is isomorphic to $\operatorname{Hom}_R(R/I, R/(\mathbf{z})) \cong \operatorname{Ext}^g_R(R/I, R)$ which is independent of $\mathbf{z}$. Thus, regardless of the choice of $\mathbf{z}$, either $\operatorname{pd}(R/I')$ is not finite or $\operatorname{pd}(R/I') = \operatorname{pd}\big(\operatorname{Ext}^g_R(R/I, R)\big) + 1$. □

2.2 Bound on the Multiplicity of Three Cubics

In this section we show that three cubic forms which (minimally) generate an ideal of height two have multiplicity at most 7. We first recall the well-known bound of 9 and then, using a formula of Huneke and Trung [HT] or Polini and Ulrich [PU] to compute the core of an ideal, we show that the correct bound for the multiplicity is 7. It is easily verified that this bound is sharp: E.g. consider the ideal $(X^3, X^2Y + XY^2, Y^3)$ in $k[X, Y]$.

Let $R = k[X_1, \ldots, X_n]$, where k is a field of characteristic 0, and let $f, g, h \in R$ be three cubic forms which minimally generate an ideal of height two. Denote by I the unmixed part of the ideal (f, g, h), which determines its multiplicity, and let $\underline{\ell} := l_1, \ldots, l_{n-2}$ be a sequence of $n - 2$ general linear forms in R. We will abuse the notation $\underline{\ell}$ to also denote the ideal generated by $l_1, \ldots, l_{n-2}$.

As I has height two, the elements $\underline{\ell}$ constitute a system of parameters of R/I. Consider the Euler characteristic

$$\begin{aligned}\chi(R/\underline{\ell}, R/I) &= \sum_{i \geqslant 0} (-1)^i \lambda\big(H_i(\underline{\ell}, R/I)\big)\\ &= \sum_{i \geqslant 0} (-1)^i \lambda\big(\operatorname{Tor}^R_i(R/\underline{\ell}, R/I)\big)\end{aligned}$$

of R/I with respect to $\underline{\ell}$. Here $\lambda(\ )$ denotes the length of a module and $H_i(\underline{\ell}, R/I)$ is the i-th Koszul homology of $\underline{\ell}$ with coefficients in R/I. By a theorem of Auslander and Buchsbaum [BH, Theorem 4.7.4] or Serre [BH, Theorem 4.7.6], one has $\chi(R/\underline{\ell}, R/I) =$

$e(\underline{\ell}; R/I)$, where e is the multiplicity symbol. If we further consider the first partial Euler characteristic

$$\chi_1(R/\underline{\ell}, R/I) = \sum_{i \geqslant 1} (-1)^{i-1} \lambda(H_i(\underline{\ell}, R/I)),$$

then we can write $e(\underline{\ell}; R/I) = \lambda(R/(\underline{\ell}, I)) - \chi_1(R/\underline{\ell}, R/I)$ in order to utilize the following non-negativity result:

Theorem 2.7 (Serre [BH, Theorem 4.7.10]). *The first partial Euler characteristic $\chi_1(R/\underline{\ell}, R/I)$ is non-negative and it is zero if and only if R/I is Cohen-Macaulay.*

So, as $\chi_1(R/\underline{\ell}, R/I) \geqslant 0$, the multiplicity of R/I is bounded above by the length of $R/(\underline{\ell}, I)$ and equality holds if and only if R/I is Cohen-Macaulay. Next we bound the length of $R/(\underline{\ell}, I)$.

Although no two elements of $\{f, g, h\}$ need form a regular sequence, we can always choose a generating set such that f and g do form a regular sequence. The Hilbert function of $R/(\underline{\ell}, f, g)$ is then given by $(1, 2, 3, 2, 1)$ and as $(f, g) \subset I$, this implies $\lambda(R/(\underline{\ell}, I)) \leqslant \lambda(R/(\underline{\ell}, f, g)) = 9$. We will now improve upon this a priori bound by exploiting the fact that I contains a third cubic h. In order to do this, we need to establish that $h \notin (\underline{\ell}, f, g)$ for a suitable choice of general linear forms $\underline{\ell}$.

Recall the notion of the *core* of an ideal as the intersection of all its (minimal) reductions — see [RS, HS]. As shown in [CPU1, Theorem 4.5], this a priori infinite intersection of ideals can be obtained by intersecting finitely many general minimal reductions. We first observe that in $\bar{R} := R/(f, g)$ the ideal $\underline{\ell}$ is a reduction of $\mathfrak{m}$ (see [NR]) with reduction number 4, that is, $\mathfrak{m}^5 \equiv \mathfrak{m}^4\, \underline{\ell}$ modulo (f, g). Indeed, given its Hilbert function above, we see that $\mathfrak{m}^5 \subseteq (\underline{\ell}, f, g)$ which implies $\mathfrak{m}^5 = \mathfrak{m}^4\, \underline{\ell} + \mathfrak{m}^2\,(f, g)$. Also note that $\overline{\mathfrak{m}} \subset \bar{R}$ is an equimultiple ideal, meaning that its height equals its analytic spread, namely $n - 2$.

Now, by definition of $\mathrm{core}(\overline{\mathfrak{m}})$, the statement that $h \notin (\underline{\ell}, f, g)$ for some choice of general linear forms $\underline{\ell}$ is equivalent to $\bar{h} \notin \mathrm{core}(\overline{\mathfrak{m}})$ in $\bar{R}$. To show that the latter condition is satisfied, we compute $\mathrm{core}(\overline{\mathfrak{m}})$ using the following formula which was conjectured in [CPU2, Conjecture 5.1] and was later established independently by Huneke-Trung and Polini-Ulrich:

Theorem 2.8 (Huneke-Trung [HT, Theorem 3.7], Polini-Ulrich [PU, Theorem 4.5])**.** *Let S be a Cohen-Macaulay local ring with residue field of characteristic zero*[1]*. Let A be an equimultiple ideal of S and let $B \subseteq A$ be a minimal reduction of A with reduction number r. Then* $\operatorname{core}(A) = B^{r+1} : A^r$.

Thus, $\operatorname{core}(\overline{\mathfrak{m}}) = {}^{(\underline{\ell}^5, f, g) : \mathfrak{m}^4}/_{(f,g)}$. To compute the colon, we first resolve $R/(\underline{\ell}^5, f, g)$ in order to determine its socle degree. Set

$$\phi := \begin{pmatrix} l_1 & \cdots & l_{n-2} & 0 & 0 & 0 & 0 \\ 0 & l_1 & \cdots & l_{n-2} & 0 & 0 & 0 \\ 0 & 0 & l_1 & \cdots & l_{n-2} & 0 & 0 \\ 0 & 0 & 0 & l_1 & \cdots & l_{n-2} & 0 \\ 0 & 0 & 0 & 0 & l_1 & \cdots & l_{n-2} \end{pmatrix},$$

and let $I_5(\phi)$ denote the ideal generated by the 5×5 minors of ϕ. Note that $I_5(\phi) = \underline{\ell}^5$ and $R/I_5(\phi)$ is perfect of grade $n-2$. Recall that $R/I_5(\phi)$ is minimally resolved by the Eagon-Northcott complex — see [EN]:

$$\mathbb{EN}(\phi): \quad 0 \to R^{\binom{n+1}{n-3}\binom{n+2}{n+2}}(-(n+2)) \xrightarrow{d_{n-2}} R^{\binom{n}{n-4}\binom{n+2}{n+1}}(-(n+1)) \xrightarrow{d_{n-3}} \cdots$$
$$\cdots \xrightarrow{d_3} R^{\binom{5}{1}\binom{n+2}{6}}(-6) \xrightarrow{d_2} R^{\binom{n+2}{5}}(-5) \xrightarrow{\wedge^5\phi} R \to R/I_5(\phi) \to 0.$$

On the other hand, $R/(f,g)$ is minimally resolved by the Koszul complex:

$$\mathbb{K}(f,g): \quad 0 \to R(-6) \to R^2(-3) \to R \to R/(f,g) \to 0.$$

Now consider the tensor product of the above complexes:

$$\mathbb{EN}(\phi) \otimes \mathbb{K}(f,g): \quad 0 \to R^{\binom{n+1}{n-3}}(-(n+8)) \to \cdots$$
$$\cdots \to R^{\binom{n+2}{5}}(-5) \oplus R^2(-3) \to R \to R/(\underline{\ell}^5, f, g) \to 0,$$

[1]This is the reason for our assumption on the characteristic of k.

and recall that $H_i(\mathbb{EN}(\phi)\otimes\mathbb{K}(f,g)) \cong \operatorname{Tor}_i^R(R/\underline{\ell}^5, R/(f,g))$ for $i \geqslant 0$ — see for example [R, Theorem 11.21]. As $\underline{\ell}$ is generated by a regular sequence and f, g form a regular sequence modulo $\underline{\ell}$, they also form a regular sequence modulo $\underline{\ell}^5$ and all higher Tor_i vanish. Therefore $\mathbb{EN}(\phi)\otimes\mathbb{K}(f,g)$ is in fact a free resolution of $R/(\underline{\ell}^5, f, g)$ of length n in which the n-th module has a twist of $-(n+8)$.

It follows that $R/(\underline{\ell}^5, f, g)$ has a pure socle generated in degree 8. To see this, we compute $\operatorname{Tor}_n^R(R/(\underline{\ell}^5, f, g), k)$ twice: On the one hand, tensoring $\mathbb{EN}(\phi)\otimes\mathbb{K}(f,g)$ with k yields

$$\operatorname{Tor}_n^R(R/(\underline{\ell}^5, f, g), k) \;\cong\; k^{\binom{n+1}{n-3}}(-(n+8)).$$

On the other hand, resolving k via the Koszul complex on the variables $X_1, \ldots, X_n$ and tensoring with $R/(\underline{\ell}^5, f, g)$ gives us

$$\operatorname{Tor}_n^R(R/(\underline{\ell}^5, f, g), k) \;\cong\; \frac{(\underline{\ell}^5, f, g) : \mathfrak{m}}{(\underline{\ell}^5, f, g)}(-n).$$

Hence $R/(\underline{\ell}^5, f, g)$ has socle isomorphic to $k^{\binom{n+1}{n-3}}(-8)$, which proves our claim.

We now show that $\operatorname{core}(\overline{\mathfrak{m}}) = {}^{(\mathfrak{m}^5,f,g)}/_{(f,g)}$. Recall that $\operatorname{core}(\overline{\mathfrak{m}}) = {}^{(\underline{\ell}^5,f,g)\,:\,\mathfrak{m}^4}/_{(f,g)}$ by Theorem 2.8, as stated above. If $r\mathfrak{m}^4 \subseteq (\underline{\ell}^5, f, g)$, then $r\mathfrak{m}^3$ is contained in the socle of $R/(\underline{\ell}^5, f, g)$ and so, modulo (f,g), the degree of r is at least 5. The reverse inclusion is clear, as $\mathfrak{m}^9 \subseteq (\underline{\ell}^5, f, g)$.

In conclusion, since h has degree 3 and $h \notin (f,g)$, we have $\bar{h} \notin \operatorname{core}(\overline{\mathfrak{m}})$ and therefore $h \notin (\underline{\ell}, f, g)$ for some choice of general linear forms $\underline{\ell}$. With this choice, the Hilbert function of $R/(\underline{\ell}, f, g, h)$ is given by $(1,2,3,1)$ and consequently $\lambda(R/(\underline{\ell}, I)) \leqslant \lambda(R/(\underline{\ell}, f, g, h)) = 7$. We summarize what we have shown:

Lemma 2.9. *Let k be a field of characteristic 0 and let $f, g, h \in R = k[X_1, \ldots, X_n]$ be three cubic forms which minimally generate an ideal of height two. Then there exist general linear forms $\underline{\ell} = l_1, \ldots, l_{n-2}$ such that $\lambda(R/(\underline{\ell}, f, g, h)) = 7$. In particular, if I denotes the unmixed part of (f, g, h), then $\lambda(R/(\underline{\ell}, I)) \leqslant 7$ and so $R/(f, g, h)$ has multiplicity at most 7.*

Lemma 2.9 is reminiscent of Chasles' version of the so-called Cayley-Bacharach Theorem. In the special case of $\mathbb{P}^2$, Chasles proved the following in 1885 — see the article of Eisenbud, Green, and Harris [EGH, Theorem CB3] for a detailed treatment and [Hu, Theorem 9.4] for a different proof:

Theorem (Chasles [C]). *Let $X_1, X_2 \subset \mathbb{P}^2$ be cubic plane curves meeting in exactly nine points. If $X \subset \mathbb{P}^2$ is any cubic containing eight of these points, then it contains the ninth point as well.*

2.3 Unmixed Ideals of Height Two

Let $R = k[X_1, \ldots, X_n]$ and denote by $\mathfrak{m}$ the homogeneous maximal ideal $(X_1, \ldots, X_n)$. In this section we give a classification of certain height two, unmixed ideals of R based on their multiplicity and number of primary components. For this purpose we recall the *associativity formula*[2] for multiplicities — see for example [BH, Corollary 4.7.8]:

$$e(R/J) \;=\; \sum_{\substack{P \,\in\, \mathrm{Spec}(R) \\ \dim(R/P) \,=\, \dim(R/J)}} e(R/P)\, \lambda(R_P/J_P). \tag{2.3}$$

Notation. With the associativity formula (2.3) in mind, in order to easily refer to certain ideals and to distinguish them by their multiplicity and their number of primary components of minimal height, we adopt the following notation: We say that an ideal J is of type

$$\langle e = a_1, \ldots, a_m; \lambda = b_1, \ldots, b_m \rangle$$

when J has m associated prime ideals of minimal height with multiplicities $a_1, \ldots, a_m$ and, locally at those primes, R/J has length $b_1, \ldots, b_m$, respectively. (So R/J has multiplicity $\sum_{i=1}^{m} a_i\, b_i$ by (2.3).)

As an immediate consequence of the associativity formula, we derive the following

Lemma 2.10. *Let $J \subset R$ be an unmixed ideal. If $I \subset R$ is an ideal containing J, with the same height and multiplicity as J, then $J = I$.*

[2] Also referred to as the *linearity formula*.

Proof. As $J \subseteq I$ and $\mathrm{ht}(J) = \mathrm{ht}(I)$, the prime ideals in the associativity formula (2.3) contributing to the multiplicity of R/I also contribute to the multiplicity of R/J. Further, the inclusion $J \subseteq I$ yields $\lambda(R_P/J_P) \geqslant \lambda(R_P/I_P)$ for $P \in \mathrm{Ass}(R/I)$. So we have the following two inequalities:

$$\begin{aligned} e(R/J) &= \sum_{\substack{P \in \mathrm{Ass}(R/J) \\ \mathrm{ht}(P)=\mathrm{ht}(J)}} e(R/P)\, \lambda(R_P/J_P) \\ &\geqslant \sum_{\substack{P \in \mathrm{Ass}(R/I) \\ \mathrm{ht}(P)=\mathrm{ht}(I)}} e(R/P)\, \lambda(R_P/J_P) \\ &\geqslant \sum_{\substack{P \in \mathrm{Ass}(R/I) \\ \mathrm{ht}(P)=\mathrm{ht}(I)}} e(R/P)\, \lambda(R_P/I_P) \\ &= e(R/I). \end{aligned}$$

As $e(R/J) = e(R/I)$, this entails that $\lambda(R_P/J_P) = \lambda(R_P/I_P)$ for all $P \in \mathrm{Ass}(R/J)$ with $\mathrm{ht}(P) = \mathrm{ht}(J)$. Due to unmixedness of J, these are all the prime ideals associated to J. So $J_P = I_P$ for all $P \in \mathrm{Ass}(R/J)$ and therefore $J = I$. □

The following lemma supplies us with a class of height two, unmixed (in fact, primary) ideals which naturally arise throughout Section 3.2. It will also be used to prove part (iv) of Proposition 2.13.

Lemma 2.11. *Let $I = (x, y)^n + (ax + by)$ with two independent linear forms x, y and elements $a, b \in R$ (of the same degree) such that $(a, b) \not\subseteq (x, y)$. Then $\mathrm{pd}(R/I) \leqslant 3$. Further, I is unmixed if and only if $\mathrm{ht}(x, y, a, b) > 3$, in which case I is primary to (x, y) and R/I has multiplicity n.*

Proof. Consider the sequence of free modules

$$0 \to R^{n-1} \xrightarrow{\varphi_3} R^{2n} \xrightarrow{\varphi_2} R^{n+2} \xrightarrow{\varphi_1} R \to 0, \tag{2.4}$$

where

$$\varphi_1 := \big(\underbrace{ax + by \quad x^n \quad x^{n-1}y \quad \cdots \quad xy^{n-1} \quad y^n}_{n+2 \text{ columns}} \big),$$

$$\varphi_2 := \underbrace{\left(\begin{array}{ccccc|ccccc} x^{n-1} & x^{n-2}y & \cdots & xy^{n-2} & y^{n-1} & & & & & \\ \hline -a & & & & & y & & & & \\ -b & -a & & & & -x & y & & & \\ & -b & \ddots & & & & -x & \ddots & & \\ & & \ddots & -a & & & & \ddots & y & \\ & & & -b & -a & & & & -x & y \\ & & & & -b & & & & & -x \end{array}\right)}_{n \text{ columns} \quad\quad n \text{ columns}} \Bigg\} n{+}2 \text{ rows},$$

and

$$\varphi_3 := \underbrace{\left(\begin{array}{ccc} y & & \\ -x & \ddots & \\ & \ddots & y \\ & & -x \\ \hline a & & \\ b & \ddots & \\ & \ddots & a \\ & & b \end{array}\right)}_{n-1 \text{ columns}} \begin{array}{l} \Big\} n \text{ rows} \\ \Big\} n \text{ rows} \end{array}$$

(The missing entries of φ_2 and φ_3 are all zero.) Note that $\varphi_1\varphi_2 = \varphi_2\varphi_3 = 0$, that is, (2.4) is a complex. Let $I_{r_i}(\varphi_i)$ denote the ideal generated by the $r_i \times r_i$ minors of φ_i, where r_i is the expected rank of φ_i. We have $r_1 = 1$, $r_2 = n+1$, and $r_3 = n-1$. Observe that

$$I_1(\varphi_1) = I, \qquad I_{n+1}(\varphi_2) = I(x,y,a,b)^{n-1}, \qquad I_{n-1}(\varphi_3) = (x,y,a,b)^{n-1}.$$

Since by assumption (x,y) is a prime ideal of height two and $(a,b) \not\subset (x,y)$, we have $\operatorname{ht}(I_{n-1}(\varphi_3)) = \operatorname{ht}(x,y,a,b) \geqslant 3$. And clearly the ideals $I_{n+1}(\varphi_2)$ and $I_1(\varphi_1)$ both have height two. Therefore, by the Buchsbaum-Eisenbud acyclicity criterion [BH, Theorem 1.4.13], the complex (2.4) is in fact a free resolution of R/I and $\operatorname{pd}(R/I) \leqslant 3$. (More precisely, $\operatorname{pd}(R/I) = 3$ if $a, b \in \mathfrak{m}$ and $\operatorname{pd}(R/I) = 2$ otherwise.)

Let $P \in \operatorname{Ass}(R/I)$ so that $\operatorname{depth}(R_P/I_P) = 0$. By the Auslander-Buchsbaum formula, $\operatorname{ht}(P) = \operatorname{pd}(R_P/I_P)$ and therefore $\operatorname{ht}(P) \leqslant 3$. Now if $\operatorname{ht}(x,y,a,b) > 3$, then $I_{n-1}(\varphi_3) \not\subset P$, while $I_n(\varphi_3) = 0$. By [BH, Lemma 1.4.9], the homomorphism φ_3 splits locally at P and $\operatorname{ht}(P) = \operatorname{pd}(R_P/I_P) = 2$. That is, I is unmixed of height two.

Conversely, if $\operatorname{ht}(x,y,a,b) = 3$, then a and b share a common divisor c modulo (x,y). Write $a \equiv ca'$ and $b \equiv cb'$ modulo (x,y) and note that the hypothesis $(a,b) \not\subset (x,y)$ forces $c \notin (x,y)$ and $(a',b') \not\subset (x,y)$. In particular, $\operatorname{ht}(x,y,c) = 3$. Multiplying $ax+by$ with $(x,y)^{n-2}$ and reducing modulo $(x,y)^n$, we obtain $c\,(a'x+b'y)\,(x,y)^{n-2} \subset I$. So $(x,y,c) \subseteq I : (a'x+b'y)(x,y)^{n-2}$, which means that (x,y,c) is contained in an associated prime of R/I. As I has height two, it cannot be unmixed.

Finally, as $\sqrt{I} = (x,y)$, I has only (x,y) as its minimal prime and it is unmixed if and only if it is primary to (x,y). Since $(a,b) \not\subset (x,y)$ and $(x,y)^n \subset I$, locally $(R/I)_{(x,y)}$ has Hilbert function $(\underbrace{1,\ldots,1}_{n \text{ times}})$ and $e(R/I) = n$ by the associativity formula (2.3). □

The following lemma, similar to Lemma 2.11, describes yet another class of unmixed ideals which we will often encounter in Section 3.2.

Lemma 2.12. *Let $I = (x^2, xy, y^2v, cx + dyv)$ with linear forms x, y, v such that $\operatorname{ht}(x, yv) = 2$ and elements $c, d \in R$ such that $\deg(c) = \deg(d) + 1$ and $(c,d) \not\subset (x,y)$. Then $\operatorname{pd}(R/I) \leqslant 3$. Further, I is unmixed if and only if $\operatorname{ht}(x,y,c,d) > 3$, in which case R/I has multiplicity 3.*

(The hypothesis $\operatorname{ht}(x,yv) = 2$ merely says that x, y as well as x, v are independent linear forms. In particular, $\operatorname{ht}(I) = 2$.)

Proof. We proceed as in the proof of Lemma 2.11. Consider the complex

$$0 \to R \xrightarrow{\varphi_3=\begin{pmatrix} c\\ d\\ y\\ x\end{pmatrix}} R^4 \xrightarrow{\varphi_2=\begin{pmatrix} -y & 0 & c & 0\\ x & -yv & dv & -c\\ 0 & x & 0 & -d\\ 0 & 0 & -x & y\end{pmatrix}} R^4 \xrightarrow{\varphi_1=(x^2\ \ xy\ \ y^2v\ \ cx+dyv)} R \to 0.$$

We have $I_1(\varphi_1) = I$ and $I_3(\varphi_2) = I\,(x,y,c,d)$, both ideals of height two, and $I_1(\varphi_3) = (x,y,c,d)$ of height at least 3. So the above complex resolves R/I and $\operatorname{ht}(P) \leqslant 3$ for any prime ideal $P \in \operatorname{Ass}(R/I)$. If $\operatorname{ht}(x, y, c, d) > 3$, then $I_1(\varphi_3) \not\subset P$ while $I_2(\varphi_3) = 0$. By [BH, Lemma 1.4.9], φ_3 splits locally at P and $\operatorname{ht}(P) = \operatorname{pd}(R_P/I_P) = 2$. That is, I is unmixed.

Conversely, assume that c and d have a common divisor $e \in \mathfrak{m}$ modulo (x, y). Write $c \equiv c'e$ and $d \equiv d'e$ modulo (x, y) and note that the condition $(c, d) \not\subset (x, y)$ implies $e \notin (x, y)$, that is, $\operatorname{ht}(x, y, e) = 3$. Reducing $cx + dyv$ modulo (x^2, xy, y^2v), we obtain $e(c'x + d'yv) \in I$ and therefore $(x, y, e) \subseteq I : (c'x + d'yv)$. As $c'x + d'yv \notin I$, this means that x, y, e are contained in some associated prime of R/I. As I has height two, it cannot be unmixed.

Lastly, assuming that I is unmixed, we determine its associated primes and compute its multiplicity using the associativity formula (2.3). Note that since $(x^2, y^2v) \subset I$, any prime ideal containing I must contain x and either y or v. In particular, we have $(x, y), (x, v) \in \operatorname{Ass}(R/I)$. But if I is unmixed, then these are the only associated primes of R/I. If $v \in (x, y)$, then I is primary to (x, y) and $e(R/I) = \lambda\big((R/I)_{(x,y)}\big) = 3$. And if $v \notin (x, y)$, then $e(R/I) = \lambda\big((R/I)_{(x,y)}\big) + \lambda\big((R/I)_{(x,v)}\big) = 2 + 1 = 3$. □

2.3.1 The Multiplicity Two Case

The following proposition identifies all height two, unmixed ideals of multiplicity 2 and provides an upper bound for their projective dimension. The critical assertion is made in part (iv) where the ideals of type $\langle e = 1; \lambda = 2\rangle$, so-called *double structures*, are classified:

Proposition 2.13. *Let $I \subset R$ be a height two, unmixed ideal of multiplicity 2. Then* $\operatorname{pd}(R/I) \leqslant 3$ *and I is one of the following:*

(i) A prime ideal generated by a linear form and an irreducible quadric.

(ii) $(x, y) \cap (x, v) = (x, yv)$ with independent linear forms x, y, v.

(iii) $(x, y) \cap (u, v) = (xu, xv, yu, yv)$ with independent linear forms x, y, u, v.

(iv) The (x,y)-primary ideal $(x,y)^2 + (ax+by)$ with independent linear forms x,y and elements $a,b \in \mathfrak{m}$ such that x,y,a,b form a regular sequence.

(iv°) (x,y^2) with independent linear forms x,y.

Proof. As $e(R/I) = 2$, by the associativity formula (2.3) I must be either of type $\langle e=2;\lambda=1\rangle$, $\langle e=1,1;\lambda=1,1\rangle$, or $\langle e=1;\lambda=2\rangle$. In the case $\langle e=2;\lambda=1\rangle$, I is a height two prime ideal of multiplicity 2. Such an ideal is generated by a linear form and an irreducible quadric. That is, I is of the form as described in part (i).

In the case $\langle e=1,1;\lambda=1,1\rangle$, I has two associated prime ideals P_1 and P_2, each of height two and multiplicity one. Say $I = Q_1 \cap Q_2$ where Q_i is P_i-primary. On the one hand, $\lambda(R_{P_i}/I_{P_i}) = 1$ implies that I equals P_i locally at P_i. On the other hand, as I is unmixed and does not have any embedded primes, we know that I equals Q_i locally at P_i. So Q_i equals P_i locally at P_i, and hence globally, and $I = P_1 \cap P_2$.

Write $P_1 = (x,y)$ and $P_2 = (u,v)$ with linear forms x,y,u,v. As P_1 and P_2 are distinct prime ideals of height two, their sum (x,y,u,v) has height strictly greater than two. And as x,y,u,v are linear forms, the height of the ideal they generate is simply the dimension of the vector space they span. So if $\operatorname{ht}(x,y,u,v) = 3$, then we can express one element in terms of the other three, say $x = \alpha y + \beta u + \gamma v$ with field coefficients α,β,γ. Subtracting αy will not change the ideal (x,y) and we may take $x = \beta u + \gamma v$. Note that this term is non-zero, as $\operatorname{ht}(x,y) = 2$. Assuming, without loss of generality, that $\beta \neq 0$, we can express u in terms of x and v. Replacing u by x does not change the ideal (u,v) and we obtain $I = (x,y) \cap (x,v) = (x,yv)$. That is, I is of the form as described in part (ii). And if $\operatorname{ht}(x,y,u,v) = 4$, then $I = (xu,xv,yu,yv)$ is of the form as described in part (iii).

Finally, in the case $\langle e=1;\lambda=2\rangle$, I is primary to a height two, prime ideal P of multiplicity one; say $P = (x,y)$ with independent linear forms x,y. As $\lambda(R_P/I_P) = 2$, we have $P^2 \subsetneq I \subsetneq P$ and I is generated by P^2 plus additional terms of the form $(a_i x + b_i y)$ with $(a_i,b_i) \not\subset (x,y)$. We claim that I contains only one such term as a minimal generator, that is, $I = (x,y)^2 + (ax+by)$ with $(a,b) \not\subset (x,y)$. To prove this, we choose one of the terms $a_ix + b_iy$ among the minimal generators of I, say $ax + by$, and first show that $\mathrm{ht}(x, y, a, b) > 3$. (That is, either a or b is a unit or x, y, a, b form a regular sequence.) Indeed, as (x, y) is prime of height two and $(a, b) \not\subset (x, y)$, the ideal (x, y, a, b) has height at least 3. If $\mathrm{ht}(x, y, a, b) = 3$, then a and b have a common divisor $c \in \mathfrak{m}$ modulo (x, y). Writing $a \equiv ca'$ and $b \equiv cb'$ modulo (x, y), we have $ax + by \equiv c(a'x + b'y)$ modulo $(x, y)^2$. As $(x, y)^2 \subset I$, the element $c(a'x + b'y)$ is a minimal generator of I and c is a zero-divisor on R/I. Since I is primary to (x, y), we must have $c \in (x, y)$. However, the condition $(a, b) \not\subset (x, y)$ implies that $c \notin (x, y)$ — a contradiction. Thus, $\mathrm{ht}(x, y, a, b) > 3$.

Our claim now follows from Lemma 2.11 which shows that the ideal $(x, y)^2+(ax+by)$ is unmixed, and Lemma 2.10, which implies that $(x, y)^2+(ax+by)$ equals I. If $a, b \in \mathfrak{m}$, then $\mathrm{ht}(x, y, a, b) = 4$ and I is of the form as described in part (iv). And if either a or b is a unit, then, after a linear change of coordinates, I is of the form as described in part (iv°).

To finish the proof, we need to verify that the projective dimension of R/I is at most 3. The ideals in parts (i), (ii), and (iv°) are complete intersections and in those cases $\mathrm{pd}(R/I) = 2$. As for part (iii), applying Lemma 2.1 to the short exact sequence

$$0 \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(x, y)} \oplus \frac{R}{(u, v)} \longrightarrow \frac{R}{(x, y, u, v)} \longrightarrow 0$$

yields $\mathrm{pd}(R/I) = 3$. And in part (iv) we have $\mathrm{pd}(R/I) = 3$ by Lemma 2.11. □

2.3.2 The Multiplicity Three Case

A classification similar to the one in Proposition 2.13 for height two, unmixed ideals of multiplicity 3 turns out to be less tractable. This is due to the difficulty of determining the ideals of type $\langle e = 1; \lambda = 3\rangle$, that is, (x, y)-primary ideas of multiplicity 3, where x, y are independent linear forms. The present body of work in Algebraic Geometry on the subject of such *nilpotent* or *multiple structures* supported on a linear subspace of codimension two seems to be confined to the generalized Cohen-Macaulay case, that is, to ideals which have locally Cohen-Macaulay quotients on the punctured spectrum — see [Ma1] and [Ma2]. For reference purposes we cite the following

Theorem (Manolache [Ma2, Theorem 1]). *The Cohen-Macaulay multiple structures of multiplicity 3 on a codimension two linear subspace $X = \mathbb{P}^n$ of $\mathbb{P}^{n+2}$, in suitable coordinates $(x_0, \ldots, x_n, x, y)$ where x, y are the generators of I_X, are the following:*

1. $(x, y)^2$,
2. (x, y^3),
3. $(x^2 + by, xy, y^2)$ *where b is a linear form in $x_0, \ldots, x_n$,*
4. $(x, y)^3 + (ax+by)(x, y) + \big(cx^2 + b^2(ax+by),\ cxy - ab(ax+by),\ cy^2 + a^2(ax+by)\big)$ *where a, b, c are forms in $x_0, \ldots, x_n$ with $\deg(a) = \deg(b) = d$ and $\deg(c) = 3d-1$ such that a, b, c have no common roots.*

To the extent of our knowledge, a classification of ideals of type $\langle e = 1; \lambda = 3\rangle$ does not currently exist. A better understanding of the structure of these ideals, particularly those with non-Cohen-Macaulay quotients, would greatly benefit our purpose by expanding the range of the methods used in Section 3.2.

To show that the above theorem fails to provide an exhaustive list in general, and to demonstrate why the method applied in the multiplicity 2 case fails for higher multiplicities, we give the following non-trivial example of such an ideal which is generated by seven cubic forms:

Example 2.14. Let $R = k[a, b, c, d, e, x, y]$ be a polynomial ring with indeterminates $a, \ldots, e, x, y$ over a field k and assume that $I \subset R$ is an ideal of multiplicity 3 and primary to (x, y). In particular, $\lambda\big((R/I)_{(x,y)}\big) = 3$ and $(x, y)^3 \subset I$. Set

$$\begin{aligned} q &:= (ac + dx)x + (bc + ey)y \\ &= (ax + by)c + dx^2 + ey^2 \end{aligned}$$

and suppose $q \in I$. As $q \notin (x, y)^2$, the Hilbert function of $(R/I)_{(x,y)}$ is given by $(1, 1, 1)$. Note that the coefficients $(ac + dx)$ and $(bc + ey)$ of x and y in q have a common divisor c modulo (x, y). Unlike the multiplicity 2 case, this does not lead to a contradiction, but merely implies that $(ax + by)x$ and $(ax + by)y$ must belong to I as well, for $q(x, y) \equiv c(ax + by)(x, y)$ modulo $(x, y)^3$ and c is a non-zerodivisor modulo I. Indeed,

$$\begin{aligned} I &= (x, y)^3 + (ax + by)(x, y) + (q) \\ &= \bigl(x^3,\, x^2y,\, xy^2,\, y^3,\, ax^2 + bxy,\, axy + by^2,\, acx + bcy + dx^2 + ey^2\bigr) \end{aligned}$$

is an ideal of type $\langle e = 1; \lambda = 3\rangle$ which is primary to (x, y) and minimally generated by seven cubic forms. One can check that $\mathrm{pd}(R/I) = 3$ and so R/I is not Cohen-Macaulay.

As a partial result, we give the following description of an (x, y)-primary ideal I of multiplicity 3 whenever $I : (x, y)$ contains a linear form l. (Note that after choosing suitable generators for (x, y), we may assume that $l = x$.)

Lemma 2.15. *If I is an ideal of multiplicity 3, primary to (x, y) with independent linear forms x, y, and $x \in I : (x, y)$, then either $I = (x, y)^2$ or $I = (x^2, xy, y^3, cx + dy^2)$ with elements c and d such that* $\mathrm{ht}(x, y, c, d) > 3$.

Proof. It is clear that $(x, y)^3 \subsetneq I$ and so we have $(x^2, xy, y^3) \subsetneq I$. In addition, as $e(R/I) = 3$, I must contain terms of the form $c_ix + b_iy + d_iy^2$ with $c_i \notin (x, y)$. Assuming $I \neq (x, y)^2$, we may choose $b_i = 0$. Indeed, multiplying $c_ix + b_iy + d_iy^2$ with y yields $b_iy^2 \in I$. As $I \neq P^2$, we have $y^2 \notin I$ and so $b_i \in (x, y)$. After reducing b_iy modulo xy and relabeling d_i, we can rewrite $c_ix + b_iy + d_iy^2$ as $c_ix + d_iy^2$.

By the same argument as in the proof of Proposition 2.13, mutatis mutandis, if $c_ix + d_iy^2$ is a minimal generator of I, then $\mathrm{ht}(x, y, c_i, d_i) > 3$. It now follows from Lemma 2.12 and Lemma 2.10 that I contains only one such term as a minimal generator, that is, $I = (x^2, xy, y^3, cx + dy^2)$ with $\mathrm{ht}(x, y, c, d) > 3$. $\square$

2.4 Varieties of Low Multiplicity

2.4.1 Varieties of Multiplicity Three (Varieties of Minimal Multiplicity)

The purpose of this section is to review the list of all (non-degenerate) homogeneous prime ideals of multiplicity 3 and to state for future reference that they all have Cohen-Macaulay quotients. In our notation, these correspond to ideals of type $\langle e = 3; \lambda = 1 \rangle$.

Let V_1 denote the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, let $V_2 \subset \mathbb{P}^4$ be any hyperplane section of V_1 which is irreducible, and let $V_3 \subset \mathbb{P}^3$ be any hyperplane section of V_2 which is irreducible, that is, a twisted cubic curve.

Theorem 2.16 (X.X.X. [X], [SD, Theorem 3])**.** *A non-degenerate, irreducible projective variety of multiplicity 3, embedded in projective space of any dimension, is either one of the following:*

(a) a cubic hypersurface,

(b) one of the varieties V_1, V_2, or V_3,

(c) a cone whose base is one of the varieties V_1, V_2, or V_3.

The defining ideal of a variety corresponding to part (a) of the above theorem is generated by an irreducible, homogeneous polynomial of degree 3 — see for example [Ha, Exercise 2.8]. As for part (c), we recall that a cone over a variety is defined by the same equations as the variety itself: Forming a (projective) cone over a variety $V \subset \mathbb{P}^n$ simply amounts to adjoining new indeterminates to the underlying polynomial ring $k[X_0, \ldots, X_n]$ and extending its defining ideal $I(V)$ to this larger ring — see for example [Ha, Exercise 2.10]. It remains part (b).

Under a suitable choice of coordinates, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$ is given by mapping $(a : b) \times (x : y : z)$ to $(ax : bx : ay : by : az : bz)$ — see for example [Ha, Exercise 2.14]. Its defining ideal as well as those of V_2 and V_3 are generated by the 2×2 minors of a 3×2 matrix of indeterminates:

$$I(V_1) = I_2 \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_5 & X_6 \end{pmatrix} \subset k[X_1, \ldots, X_6],$$

$$I(V_2) = I_2 \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_4 & X_5 \end{pmatrix} \subset k[X_1, \ldots, X_5],$$

$$I(V_3) = I_2 \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \\ X_3 & X_4 \end{pmatrix} \subset k[X_1, \ldots, X_4].$$

We see that all three ideals $I(V_1)$, $I(V_2)$, and $I(V_3)$ have height two and, by the Hilbert-Burch Theorem, their quotients are Cohen-Macaulay. (In particular, these are prime ideals of *minimal multiplicity* — see for example [EH].) We also note that each of these ideals is generated by three quadric forms in no more than 6 indeterminates.

2.4.2 Varieties of Multiplicity Four

In this section we draw on a result of Swinnerton-Dyer to study all (non-degenerate) homogeneous prime ideals of multiplicity 4. In our notation, these correspond to ideals of type $\langle e = 4; \lambda = 1 \rangle$.

Let V_4 denote the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$ and let V_5 denote the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Theorem 2.17 (Swinnerton-Dyer [SD, Theorem 1])**.** *A non-degenerate, irreducible projective variety of multiplicity 4, embedded in projective space of any dimension, is either one of the following:*

(a) a quartic hypersurface,

(b) the complete intersection of two quadric hypersurfaces,

(c) *one of the varieties* V_4 *or* V_5*,*

(d) *a variety obtained from* V_4 *or* V_5 *by a succession of hyperplane sections and/or projections from a point onto a hyperplane,*

(e) *a cone whose base is one of the varieties described in part (c) or (d).*

Under a suitable choice of coordinates, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ into $\mathbb{P}^7$ is given by sending $(x:y) \times (p:q:r:s)$ to $(px:qx:rx:sx:py:qy:ry:sy)$. Its image V_4 can be regarded as the (projectivized) set of all 2×4 matrices of rank 1. Thus, its defining ideal is generated by the 2×2 minors of a generic 2×4 matrix:

$$I(V_4) = I_2 \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix} \subset k[X_{11}, \ldots, X_{24}].$$

And the Veronese surface, the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, is the image of the map $(x:y:z) \mapsto (x^2:y^2:z^2:xy:yz:xz)$ whose defining ideal is generated by the 2×2 minors of a symmetric 3×3 matrix of indeterminates:

$$I(V_5) = I_2 \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{pmatrix} \subset k[X_1, \ldots, X_6].$$

We note that both ideals $I(V_4)$ and $I(V_5)$ have height three, their quotients are Cohen-Macaulay, and both are generated by 6 quadric forms in no more than 8 indeterminates.

2.4.3 The Codimension Two Case (Varieties of Almost Minimal Multiplicity)

Throughout this dissertation we are primarily concerned with ideals of height two, in particular ideals with non-Cohen-Macaulay quotients. Since the defining ideals of V_4 and V_5 introduced above both have height three, and the same holds for any hyperplane sections of V_4 and V_5, it follows from Theorem 2.17 that any homogeneous prime ideal of height two and multiplicity 4 which is not a complete intersection must arise from a projection of either V_4 or V_5 onto a hyperplane.

To get a glimpse of such an ideal, we carry out such a projection computationally for each of the varieties V_4 and V_5 using the software Macaulay 2 [M2]. The commands listed below calculate (the ideals of) the projections of $V_4 \subset \mathbb{P}^7$ and $V_5 \subset \mathbb{P}^5$ from a general point of their ambient space onto $\mathbb{P}^6$ and $\mathbb{P}^4$, respectively. (For a description of these commands and more, we refer the reader to [EGSS].)

We are largely interested in two aspects of these computations: The minimal graded free resolution of the resulting ideals — in particular their projective dimension — and the degrees of their generators. Both of these are data which we read off from the corresponding Betti diagrams at the end of each computation.

First we compute a generic projection of the Segre variety $V_4 \subset \mathbb{P}^7$.

```
Macaulay 2, version 0.9.2-20
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : R = QQ[x_0..x_7];

i2 : genericMatrix(R, x_0, 2, 4)

o2 = | x_0 x_2 x_4 x_6 |
     | x_1 x_3 x_5 x_7 |

             2       4
o2 : Matrix R  <--- R

i3 : segre = trim minors(2, o2)

o3 = ideal (x x  - x x , x x  - x x , x x  - x x , x x  - x x , ... )
             5 6    4 7   3 6    2 7   1 6    0 7   3 4    2 5

o3 : Ideal of R

i4 : degree segre == 4

o4 = true

i5 : codim segre == 3

o5 = true

i6 : Rbar = R/segre;
```

```
i7 : point = random(Rbar^{1}, Rbar^7)

o7 = {-1} | 5/2x_1+x_2-8/9x_4-2/5x_5-2/3x_6 ... |

                1          7
o7 : Matrix Rbar  <--- Rbar

i8 : S = QQ[y_0..y_6];

i9 : proj = kernel map(Rbar, S, point)

             2   266385983199364652816500943
o9 = ideal (y  - ---------------------------*y y  - ... )
             0   791467040466942314474149731  0 1

o9 : Ideal of S

i10 : degree proj == 4

o10 = true

i11 : codim proj == 2

o11 = true

i12 : betti res proj

o12 = total: 1 4 4 1
          0: 1 . . .
          1: . 1 . .
          2: . 3 4 1
```

The result is a prime ideal of height two and multiplicity 4 which has projective dimension 3 and is generated by three cubics and a quadric.

Next we compute a generic projection of the Veronese variety $V_5 \subset \mathbb{P}^5$.

```
Macaulay 2, version 0.9.2-20
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : R = QQ[x_0..x_5];

i2 : genericSymmetricMatrix(R, x_0, 3)
```

```
o2 = | x_0 x_1 x_2 |
     | x_1 x_3 x_4 |
     | x_2 x_4 x_5 |

             3       3
o2 : Matrix R  <--- R

i3 : veronese = trim minors(2, o2)

             2                                    2
o3 = ideal (x  - x x , x x  - x x , x x  - x x , x  - x x , x x  - x x , ... )
             4    3 5   2 4    1 5   2 3    1 4   2    0 5   1 2    0 4

o3 : Ideal of R

i4 : degree veronese == 4

o4 = true

i5 : codim veronese == 3

o5 = true

i6 : Rbar = R/veronese;

i7 : point = random(Rbar^{1}, Rbar^5)

o7 = {-1} | 5/2x_1+x_2-8/9x_4-2/5x_5 -2/3x_0-9/5x_2-3/2x_3-4/3x_4-7/4x_5 ... |

                1          5
o7 : Matrix Rbar  <--- Rbar

i8 : S = QQ[y_0..y_4];

i9 : proj = kernel map(Rbar, S, point)

             2     1188901803024280156
o9 = ideal (y y  - -------------------*y y y  - ... )
             0 3    929813254718426775  0 1 3

o9 : Ideal of S

i10 : degree proj == 4

o10 = true

i11 : codim proj == 2

o11 = true

i12 : betti res proj
```

```
o12 = total: 1 7 10 5 1
          0: 1 .  . . .
          1: . .  . . .
          2: . 7 10 5 1
```

We obtain a prime ideal of height two and multiplicity 4 which has projective dimension 4 and is generated by seven cubics.

All examples of height two, multiplicity 4 prime ideals computed in this way yield the same Betti diagrams displayed above. As it turns out, this is no coincidence: Recent work of Brodmann and Schenzel [BSCH] on varieties of *almost minimal multiplicity* shows that these graded Betti numbers are uniquely determined by our assumptions on height and multiplicity:

Theorem 2.18 (Brodmann-Schenzel [BSCH, Theorem 2.1]). *A non-degenerate, irreducible projective variety V of multiplicity 4 and codimension 2, which is not a cone, is one of the following:*

(a) the complete intersection of two quadric hypersurfaces,

(b) $\dim V \leqslant 4$ *and the Betti diagram of* $I(V)$ *has the form*

	0	1	2	3
0	1	–	–	–
1	–	1	–	–
2	–	3	4	1

(c) (The exceptional case) V *is a generic projection of the Veronese surface* $V_5 \subset \mathbb{P}^5$ *and the Betti diagram of* $I(V_5)$ *has the form*

	0	1	2	3	4
0	1	–	–	–	–
1	–	–	–	–	–
2	–	7	10	5	1

Moreover, for any $d = 1 \ldots 4$ *there are examples as mentioned in part (b) such that* $\dim V = d$.

Chapter 3

Three-generated Ideals

3.1 The Unmixed Part and Linkage

The following result is contained in the paper of Huneke and Ulrich [HU2] in the context of *residual intersections* and was later generalized by Fan in a more elementary setting using homological algebra.

Theorem 3.1 (Huneke-Ulrich [HU2], Fan [F, Corollary 1.2])**.** *Let R be a regular local ring and let J be a three-generated ideal of R with* $\operatorname{ht}(J) = 2$*. Let I denote the unmixed part of J. If R/I is Cohen-Macaulay, i.e.* $\operatorname{pd}(R/I) = 2$*, then* $\operatorname{pd}(R/J) \leqslant 3$.

By virtue of Theorem 3.1, in our effort to bound the projective dimension of R/J, we may restrict our attention to those ideals J whose unmixed parts I are non-degenerate:

Corollary 3.2. *Let R be a regular local ring and let J be a three-generated ideal of R with* $\operatorname{ht}(J) = 2$*. If the unmixed part of J contains a linear form, then* $\operatorname{pd}(R/J) \leqslant 3$.

Proof. Let I denote the unmixed part of J and let $l \in I$ be a linear form. Then $I/(l)$ is a height one, unmixed ideal of the unique factorization domain $R/(l)$, and therefore principal. Lifting a generator of $I/(l)$ back to R, along with l, gives a generating set for the ideal I. As $\operatorname{ht}(I) = 2$, R/I is a complete intersection and $\operatorname{pd}(R/J) \leqslant 3$ by Theorem 3.1. □

We now extend Theorem 3.1 in the following sense: We remove the Cohen-Macaulay assumption on R/I and give a bound for the projective dimension of R/J in terms of the projective dimension of an ideal linked to I. (See Section 2.1.3 for the definition of linkage.) As it turns out, this bound is sharp unless $\operatorname{pd}(R/J) \leqslant 3$.

Theorem 3.3. *Let R be a regular local ring and let J be a three-generated ideal of R with $\operatorname{ht}(J) = 2$. Denote by I the unmixed part of J and let $a, b \in I$ be a regular sequence. Then*

$$\operatorname{pd}(R/J) \leqslant \operatorname{pd}\left({}^{R}\!/\!_{(a,b):I}\right) + 1$$

and equality holds if $\operatorname{pd}(R/J) \geqslant 4$.

Proof. Let M and N denote second syzygy modules of R/J and R/I, respectively. That is, there are short exact sequences

$$0 \to M \to F \to J \to 0, \tag{3.1}$$

$$0 \to N \to G \to I \to 0 \tag{3.2}$$

with free modules $F \cong R^3$ and G. We first exhibit a bound for the projective dimension of $M^* = \operatorname{Hom}_R(M, R)$ in terms of $\operatorname{pd}\left({}^{R}\!/\!_{(a,b):I}\right)$. Then, by drawing on a result of Bruns [B2] on (oriented) second syzygy modules, we will establish a bound for the projective dimension of R/J.

In the proof of [F, Theorem 1.1], Fan establishes a short exact sequence

$$0 \to G^* \to F^* \oplus N^* \to M^* \to 0$$

as follows: Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \big\downarrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & I & \longrightarrow & 0
\end{array}$$

in which the exact rows are given by (3.1) and (3.2) and the vertical maps are induced by the inclusion $J \overset{\iota}{\hookrightarrow} I$. By dualizing the above diagram, one obtains the commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & I^* & \longrightarrow & G^* & \longrightarrow & N^* & \xrightarrow{\Delta_I} & \operatorname{Ext}^1_R(I,R) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \iota^*} & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Ext}^1_R(\iota,R)} & & \\
0 & \longrightarrow & J^* & \longrightarrow & F^* & \longrightarrow & M^* & \xrightarrow{\Delta_J} & \operatorname{Ext}^1_R(J,R) & \longrightarrow & 0
\end{array}
$$

with exact rows and connecting homomorphisms Δ_I, Δ_J. The mapping cone of this diagram is the exact sequence

$$0 \to I^* \to J^* \oplus G^* \to F^* \oplus N^* \to M^* \oplus \operatorname{Ext}^1_R(I,R) \to \operatorname{Ext}^1_R(J,R) \to 0. \tag{3.3}$$

As I is the unmixed part of J and $\operatorname{ht}(J) = \operatorname{ht}(I) = 2$, we have $(I/J)_P = 0$ locally at every prime ideal P of height at most 2. So $\operatorname{grade}(\operatorname{ann}(I/J)) = \operatorname{ht}(\operatorname{ann}(I/J)) \geqslant 3$ and therefore, by the Ext characterization of grade, we have $\operatorname{Ext}^i_R(I/J,R) = 0$ for $i = 0,1,2$. Now the long exact sequence on $\operatorname{Ext}^\bullet_R(-,R)$ induced by the short exact sequence $0 \to J \xrightarrow{\iota} I \to I/J \to 0$ yields that both homomorphisms

$$\iota^* : I^* \xrightarrow{\sim} J^* \quad \text{and} \quad \operatorname{Ext}^1_R(\iota,R) : \operatorname{Ext}^1_R(I,R) \xrightarrow{\sim} \operatorname{Ext}^1_R(J,R)$$

are indeed isomorphisms. Thus, the first and the last map in (3.3) split and we obtain the desired short exact sequence:

$$0 \to G^* \to F^* \oplus N^* \to M^* \to 0. \tag{3.4}$$

Note that F^* and G^* are free modules and have projective dimension 0. So, by Lemma 2.1, the short exact sequence (3.4) implies $\operatorname{pd}(M^*) \leqslant \max\{1, \operatorname{pd}(N^*)\}$. Next we bound the projective dimension of N^*.

As $\operatorname{grade}(I) = 2$, we have $\operatorname{Ext}^1_R(R/I,R) = 0$ and the long exact sequence on $\operatorname{Ext}^\bullet_R(-,R)$ induced by the short exact sequence $0 \to I \to R \to R/I \to 0$ yields $I^* \cong R$. The same long exact sequence also gives us $\operatorname{Ext}^1_R(I,R) \cong \operatorname{Ext}^2_R(R/I,R)$. So dualizing (3.2) gives rise to the exact sequences

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \longrightarrow & G^* & \longrightarrow & & \longrightarrow & N^* & \longrightarrow & \operatorname{Ext}^2_R(R/I,R) \longrightarrow 0, \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & K & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & &
\end{array}
$$

where $K = \operatorname{Im}(G^* \to N^*)$. As $0 \to R \to G^* \to K \to 0$ is a free resolution of K, $\operatorname{pd}(K) \leqslant 1$. Hence, by Lemma 2.1, $\operatorname{pd}(N^*) \leqslant \max\{1, \operatorname{pd}(\operatorname{Ext}^2_R(R/I, R))\}$.

In order to bound the projective dimension of $\operatorname{Ext}^2_R(R/I, R)$, recall that for any regular sequence $a, b \in I$, $\operatorname{Ext}^2_R(R/I, R) \cong \operatorname{Hom}_R(R/I, R/(a, b)) \cong {}^{(a,b):I}/_{(a,b)}$ — see for example [BH, Lemma 1.2.4]. Yet another application of Lemma 2.1 to the short exact sequence

$$0 \longrightarrow \frac{(a,b):I}{(a,b)} \longrightarrow \frac{R}{(a,b)} \longrightarrow \frac{R}{(a,b):I} \longrightarrow 0 \tag{3.5}$$

delivers $\operatorname{pd}\big({}^{(a,b):I}/_{(a,b)}\big) \leqslant \max\{2, \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) - 1\}$. Combining the inequalities obtained so far, we arrive at $\operatorname{pd}(M^*) \leqslant \max\{2, \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) - 1\}$.

As J is a three-generated ideal, the free module F in the short exact sequence (3.1) has rank three, whence M is a second syzygy module of rank two. A result of Bruns [B2, Corollary 2.6] now states that $M \cong M^*$. So $\operatorname{pd}(R/J) = \operatorname{pd}(M^*) + 2$ and that gives us

$$\operatorname{pd}(R/J) \leqslant \max\{4, \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) + 1\}\,. \tag{3.6}$$

Note that $\operatorname{pd}\big({}^{R}/_{(a,b):I}\big) \geqslant \operatorname{grade}((a,b):I) = 2$. If $\operatorname{pd}\big({}^{R}/_{(a,b):I}\big) = 2$, then ${}^{R}/_{(a,b):I}$ is Cohen-Macaulay and therefore so is R/I. In this case Theorem 3.1 asserts that $\operatorname{pd}(R/J) \leqslant 3 = \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) + 1$, as claimed. And if $\operatorname{pd}\big({}^{R}/_{(a,b):I}\big) \geqslant 3$, then the desired inequality $\operatorname{pd}(R/J) \leqslant \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) + 1$ follows directly from (3.6).

It remains to show that $\operatorname{pd}(R/J) = \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) + 1$ whenever $\operatorname{pd}(R/J) \geqslant 4$. Set $j := \operatorname{pd}(R/J)$ and assume that $j \geqslant 4$. So we have $\operatorname{pd}(M^*) = \operatorname{pd}(M) = j - 2 \geqslant 2$. Applying Lemma 2.1 to the short exact sequence (3.4) yields $\operatorname{pd}(N^*) = \operatorname{pd}(M^*)$. As $\operatorname{pd}(K) \leqslant 1$ and $\operatorname{pd}(N^*) \geqslant 2$, Lemma 2.1 and the short exact sequence

$$0 \to K \to N^* \to {}^{(a,b):I}/_{(a,b)} \to 0$$

imply that $\operatorname{pd}\big({}^{(a,b):I}/_{(a,b)}\big) = \operatorname{pd}(N^*) = j - 2$. Finally, by the short exact sequence (3.5), $\operatorname{pd}\big({}^{R}/_{(a,b):I}\big) \leqslant \max\{j-1, 2\} = j - 1$, that is, $\operatorname{pd}(R/J) \geqslant \operatorname{pd}\big({}^{R}/_{(a,b):I}\big) + 1$. This finishes the proof, as we have already established the reverse inequality above. $\square$

Theorem 3.3 proves to be useful even in instances where we cannot determine the unmixed part I, but where we can choose elements $a, b \in I$ of sufficiently low degree in order to bound the multiplicities of the ideals I and $(a,b) : I$ and use this information to infer the existence of an upper bound for $\mathrm{pd}\big({}^{R}/_{(a,b):I}\big)$ and consequently for $\mathrm{pd}(R/J)$ — see proof of Theorem 3.9 for a concrete application.

As stated, Theorem 3.3 raises two questions which are not answered by the proof given above:

Clearly, equality cannot hold in $\mathrm{pd}(R/J) \leqslant \mathrm{pd}\big({}^{R}/_{(a,b):I}\big) + 1$ if $\mathrm{pd}(R/J) = 2$. If $\mathrm{pd}(R/J) = 2$, that is, if R/J is Cohen-Macaulay, then J equals its unmixed part I and the link ${}^{R}/_{(a,b):I}$ is Cohen-Macaulay as well: $\mathrm{pd}(R/J) = \mathrm{pd}\big({}^{R}/_{(a,b):I}\big) = 2$. But what about the case $\mathrm{pd}(R/J) = 3$? In other words, is the link ${}^{R}/_{(a,b):I}$ then necessarily Cohen-Macaulay? (This would constitute the converse of Theorem 3.1.)

Secondly, how essential is the hypothesis that J is a three-generated ideal? A priori, this is a rather crucial assumption which allows us to apply the result of Bruns [B2] and to establish that $M \cong M^*$. But does the conclusion of Theorem 3.3 hold for any *almost complete intersection*, that is, for an N-generated ideal of height $N-1$?

These considerations led to a simpler proof and to a generalization of Theorem 3.3 which answers the above questions in the positive:

Proposition 3.4. *Let R be a regular local ring and let J be an N-generated ideal of R with $\mathrm{ht}(J) = N-1$. If $\mathbf{z} = z_1, \ldots, z_{N-1}$ is a regular sequence in J, then*

$$\mathrm{pd}(R/J) \leqslant \mathrm{pd}\big({}^{R}/_{(\mathbf{z}):J}\big) + 1$$

and equality holds if and only if R/J is not Cohen-Macaulay, i.e. iff $\mathrm{pd}(R/J) \geqslant N$.

Proof. Let $J = (f_1, \ldots, f_N)$ with $\mathrm{ht}(f_1, \ldots, f_{N-1}) = N-1$ and let $\mathbf{z}$ be a maximal regular sequence in J. Recall that by Lemma 2.3 we have $(\mathbf{z}) : J = (\mathbf{z}) : J^{\mathrm{unm}}$. That is, $(\mathbf{z}) : J$ is linked to the unmixed part of J. By Lemma 2.6, any two links of J^{unm} have the same projective dimension; in particular, $\mathrm{pd}\big({}^{R}/_{(\mathbf{z}):J^{\mathrm{unm}}}\big) = \mathrm{pd}\big({}^{R}/_{(f_1,\ldots,f_{N-1}):J^{\mathrm{unm}}}\big)$. So it suffices to prove the claim for $\mathbf{z} = f_1, \ldots, f_{N-1}$.

Note that $(f_1, \ldots, f_{N-1}) : J = (f_1, \ldots, f_{N-1}) : f_N$. That gives us the short exact sequence

$$0 \longrightarrow \frac{R}{(f_1, \ldots, f_{N-1}) : J} \xrightarrow{\cdot f_N} \frac{R}{(f_1, \ldots, f_{N-1})} \longrightarrow \frac{R}{J} \longrightarrow 0,$$

of which the middle term $R/(f_1, \ldots, f_{N-1})$ is resolved by the Koszul complex on the elements $f_1, \ldots, f_{N-1}$ and has projective dimension $N-1$. Since $\mathrm{pd}\big({}^{R}/_{(f_1,\ldots,f_{N-1}):J}\big) \geqslant \mathrm{grade}((f_1, \ldots, f_{N-1}) : J) = N-1$, it follows from Lemma 2.1 that $\mathrm{pd}(R/J) \leqslant \mathrm{pd}\big({}^{R}/_{(f_1,\ldots,f_{N-1}):J}\big) + 1$, as claimed.

Now if R/J is not Cohen-Macaulay, then $\mathrm{pd}(R/J) \geqslant N$ and we also have the reverse inequality $\mathrm{pd}\big({}^{R}/_{(f_1,\ldots,f_{N-1}):J}\big) \leqslant \mathrm{pd}(R/J) - 1$ by Lemma 2.1. And if R/J is Cohen-Macaulay, then J is unmixed and $(\mathbf{z}) : J$ is linked to J by part (a) of Proposition 2.4. By part (b) of Proposition 2.4 ${}^{R}/_{(\mathbf{z}):J}$ is Cohen-Macaulay as well and $\mathrm{pd}(R/J) = \mathrm{pd}\big({}^{R}/_{(\mathbf{z}):J}\big) = N-1$ by the Auslander-Buchsbaum formula. □

3.2 The Projective Dimension of Three Cubics

Let $R = k[X_1, \ldots, X_n]$, where k is a field of characteristic 0, and let $f, g, h \in R$ be three cubic forms. Our goal in this section is to give a bound, independent of n, for the projective dimension of $R/(f, g, h)$. Clearly, we may assume that the elements f, g, h minimally generate the ideal (f, g, h), as otherwise the bound $\mathrm{pd}(R/(f, g, h)) \leqslant 2$ follows immediately. We first reduce the problem to the case where the ideal (f, g, h) has height two.

By Krull's (generalized) Principal Ideal Theorem, $\mathrm{ht}(f, g, h) \leqslant 3$. If $\mathrm{ht}(f, g, h) = 1$, then (f, g, h) is contained in a prime ideal of height one. As R is a unique factorization domain, this prime ideal is principal. So the elements f, g, h share a common divisor and (f, g, h) is isomorphic to an ideal generated either by three quadrics, in which case $\mathrm{pd}(R/(f, g, h)) \leqslant 4$, or by three linear forms, in which case $\mathrm{pd}(R/(f, g, h)) \leqslant 3$. And if $\mathrm{ht}(f, g, h) = 3$, then f, g, h form a regular sequence and $\mathrm{pd}(R/(f, g, h)) \leqslant 3$. So, without loss of generality, $\mathrm{ht}(f, g, h) = 2$.

By Lemma 2.9, the multiplicity of $R/(f,g,h)$ is at most 7. In what follows, we prove the existence of a bound for the projective dimension of $R/(f,g,h)$ by way of a case-by-case analysis for each possible value of the multiplicity of $R/(f,g,h)$, that is, for $e(R/(f,g,h)) = 1 \dots 7$.

Throughout Section 3.2 we shall denote by I the unmixed part of the ideal (f,g,h). So, in particular, $\mathrm{ht}(I) = \mathrm{ht}(f,g,h) = 2$ and both ideals have the same multiplicity. We summarize what we will prove in the forthcoming Sections 3.2.1 through 3.2.7.

Theorem 3.5. *If f,g,h are three cubic forms in a polynomial ring R over a field of characteristic zero, then $\mathrm{pd}(R/(f,g,h)) \leqslant 36$. More precisely, with $I = (f,g,h)^{\mathrm{unm}}$,*

(a) If $\mathrm{ht}(f,g,h) = 2$ and I contains a linear form, or if $\mathrm{ht}(f,g,h) = 3$, then $\mathrm{pd}(R/(f,g,h)) \leqslant 3$. (See Corollary 3.2 in Section 3.1.)

(b) If I contains a quadric, or if $\mathrm{ht}(f,g,h) = 1$, then $\mathrm{pd}(R/(f,g,h)) \leqslant 4$. (See Theorem 3.9 in Section 3.2.4.)

(c) Suppose $e(R/(f,g,h)) = 3$, 4, or 5 and let I' be an ideal which is linked to I by a complete intersection generated by cubics. If I' contains a quadric, then $\mathrm{pd}(R/(f,g,h)) \leqslant 4$. (See Theorem 3.12 in Section 3.2.6.)

(d) Below we give a breakdown of the bounds by multiplicity.

multiplicity of $R/(f,g,h)$	bound on $\mathrm{pd}(R/(f,g,h))$
$1, 7$	3
2	4
3	16
4	36
$5, 6$	20

The above bounds of 16, 36, and 20 in the cases of multiplicity 3 through 6 are most likely not optimal. The bound of 16 is derived by showing that the cubics f, g, h are contained in an ideal generated by four quadrics in four variables, which entails that f, g, h can be expressed entirely in terms of 16 linear forms. The bound of 36 stems from Proposition 3.11 in Section 3.2.5, while the bound of 20 is due to the argument of Remark 3.13 in Section 3.2.6.

There are no known examples of three cubics f, g, h with $\mathrm{pd}(R/(f, g, h))$ greater than 5. An example with $\mathrm{pd}(R/(f, g, h)) = 5$ is constructed in Section 3.3.

3.2.1 Multiplicities One and Seven

In the case of multiplicity one, the optimal bound for the projective dimension of $R/(f, g, h)$ is readily verified to be 3. Namely, by the associativity formula (2.3), the unmixed part I is of type $\langle e = 1; \lambda = 1\rangle$. (See page 24 in Section 2.3 for explanation of this notation.) So I is primary to a height two, prime ideal P of multiplicity one. Such a prime ideal is generated by two linear forms. Since $\lambda(R_P/I_P) = 1$ and I is P-primary, we have $I = P$ and $\mathrm{pd}(R/(f, g, h)) \leqslant 3$ by Corollary 3.2.

In the case of multiplicity 7, we first show that R/I is again Cohen-Macaulay. Recall that (regardless of multiplicity) there exist $n-2$ general linear forms $\underline{\ell} = l_1, \ldots, l_{n-2}$ such that $e(R/I) \leqslant \lambda(R/(\underline{\ell}, I))$ by Theorem 2.7 and $\lambda(R/(\underline{\ell}, I)) \leqslant 7$ by Lemma 2.9. Now, if $e(R/I) = 7$, then we have $e(R/I) = \lambda(R/(\underline{\ell}, I))$ and R/I is Cohen-Macaulay by Theorem 2.7. By Theorem 3.1, $\mathrm{pd}(R/(f, g, h)) \leqslant 3$.

3.2.2 Multiplicity Two

Using Proposition 2.13, we establish the bound of 4 for the projective dimension of $R/(f, g, h)$ in the case of multiplicity 2. If I contains a linear form, as in parts (i), (ii), and (iv°) of the proposition, then $\mathrm{pd}(R/(f, g, h)) \leqslant 3$ by Corollary 3.2.

If I is of the form as in part (iii), that is, $I = (xu, xv, yu, yv)$ with independent linear forms x, y, u, v, then Theorem 3.1 does not apply as R/I is not Cohen-Macaulay.

(If R/I were Cohen-Macaulay, then $\operatorname{pd}(R/I) = \operatorname{ht}(I) = 2$ by the Auslander-Buchsbaum formula. By the Hilbert-Burch Theorem, I would then be generated by forms of degree at least 3, namely by the 3×3 minors of a 4×3 matrix. A contradiction, as I contains forms of degree 2.) Here we choose the regular sequence $xu, yv \in I$ to compute the link $(xu, yv) : I = (xy, xu, vy, vu)$, which has projective dimension 3. So $\operatorname{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.

The following example shows that the bound of 4 obtained in this way is optimal:

Example 3.6. Let $R = k[l_{ij}, x, y, u, v]$ with $i = 1 \dots 3$, $j = 1 \dots 4$, and set

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{pmatrix} \begin{pmatrix} xu \\ xv \\ yu \\ yv \end{pmatrix}.$$

Then $R/(f, g, h)$ has the minimal free resolution

$$0 \to R(-9) \to R^4(-8) \to \begin{matrix} R^3(-6) \\ \bigoplus \\ R^2(-7) \end{matrix} \to R^3(-3) \to R \to R/(f, g, h) \to 0.$$

It remains part (iv) of Proposition 2.13 where $I = (x, y)^2 + (ax + by)$ and the elements $x, y, a, b \in \mathfrak{m}$ form a regular sequence. Here we compute the link $(x^2, y^2) : I$ in order to apply Theorem 3.3. We claim that

$$(x^2, y^2) : I = (x, y)^2 + (ax - by).$$

To show this, we first observe that $(x, y)^2 + (ax - by) \subseteq (x^2, y^2) : I$, which amounts to checking $(xy, ax - by)(xy, ax + by) \subseteq (x^2, y^2)$. The ideal $(x, y)^2 + (ax - by)$ is clearly unmixed of height two and has multiplicity 2. (Compare it with the ideal I or simply apply Lemma 2.11.) Next we note that by Macaulay's Unmixedness Theorem, the ideal (x^2, y^2) is unmixed and therefore $\operatorname{ht}\big((x^2, y^2) : I\big) = \operatorname{ht}\big(x^2, y^2\big) = 2$. (We do not use the fact that $(x^2, y^2) : I$ is unmixed as well.) Further, the multiplicity $e\big({}^{R}/_{(x^2,y^2):I}\big)$ of the link is given by $e\big(R/(x^2, y^2)\big) - e(R/I) = 4 - 2 = 2$. By Lemma 2.10, $(x^2, y^2) : I = (x, y)^2 + (ax - by)$, as claimed.

We can now apply Theorem 3.3. By Lemma 2.11, $\mathrm{pd}\left({}^{R}\!/_{(x,y)^2+(ax-by)}\right) \leqslant 3$ and so $\mathrm{pd}(R/(f,g,h)) \leqslant 4$. The following example, similar to Example 3.6 above, shows that the bound of 4 obtained in this way is optimal:

Example 3.7. Let $R = k[l_{ij}, a, b, x, y]$ with $i = 1 \ldots 3$, $j = 1 \ldots 4$, and set

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \\ ax+by \end{pmatrix}.$$

Then $R/(f,g,h)$ has the same graded Betti numbers as in Example 3.6. In particular, $\mathrm{pd}(R/(f,g,h)) = 4$.

3.2.3 Multiplicity Three

In the case of multiplicity 3, we give a bound of 16 for the projective dimension of $R/(f,g,h)$. By the associativity formula (2.3) there are five cases to consider:

$\underline{\langle e = 3; \lambda = 1\rangle}$ I is a homogeneous prime ideal of *minimal multiplicity*, that is, $e(R/I) = \mathrm{ht}(I) + 1$. (Recall that by Corollary 3.2 we may assume I is non-degenerate.) As shown in Section 2.4.1, R/I is Cohen-Macaulay and we have $\mathrm{pd}(R/(f,g,h)) \leqslant 3$ by Theorem 3.1.

$\underline{\langle e = 1; \lambda = 3\rangle}$ I is primary to $P = (x,y)$ with independent linear forms x, y and the Hilbert function of $(R/I)_P$ is either $(1,2)$ or $(1,1,1)$. That is, locally at P, the ideal I is either of the form $(x,y)_P^2$ or of the form $(x,y)_P^3 + (cx+dy)_P$ with $(c,d) \not\subseteq (x,y)$.

In the former case we must have $I \subseteq P^2$, as otherwise $(R/I)_P$ would have Hilbert function $(1,1,1)$. But $\mathrm{Ass}(R/I) = \{P\}$ and $I_P = P_P^2$. So $I = P^2$ globally and $\mathrm{pd}(R/(f,g,h)) \leqslant 3$ by Theorem 3.1.

Now suppose $(R/I)_P$ has Hilbert function $(1,1,1)$. Note that $I : P$ is also primary to P and has multiplicity 2. By parts (iv) and (iv°) of Proposition 2.13 we have either $I : P = (x,y)^2 + (ax+by)$ where x, y, a, b form a regular sequence, or $I : P = (x, y^2)$.

If $I : P = (x, y^2)$, then the mere fact that $I : P$ contains a linear form allows us to give an explicit description of I in terms of its generators: By Lemma 2.15, $I = (x^2, xy, y^3, cx + dy^2)$ with elements c and d such that $\mathrm{ht}(x, y, c, d) > 3$. (We are assuming $I \neq (x, y)^2$.) Having determined I explicitly, we are able compute the link $(x^2, y^3) : I$ in order to apply Theorem 3.3. We claim that

$$(x^2, y^3) : I = (x^2, xy, y^3, cx - dy^2).$$

To show this, we first observe that $(x^2, xy, y^3, cx - dy^2) \subseteq (x^2, y^3) : I$, which amounts to checking $(xy, cx - dy^2)(xy, cx + dy^2) \subseteq (x^2, y^3)$. Next we note that the ideal $(x^2, xy, y^3, cx - dy^2)$ is unmixed of height two and multiplicity 3. (Compare it with the ideal I or simply apply Lemma 2.12.) Clearly, the ideal $(x^2, y^3) : I$ has height two and multiplicity 3 as well. So our claim follows from Lemma 2.10. Now, by Lemma 2.12, $\mathrm{pd}\big(R/(x^2, xy, y^3, cx - dy^2)\big) \leqslant 3$ and so $\mathrm{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.

For the remaining case $I : P = P^2 + (ax + by)$ we first note that $\deg(ax + by) \leqslant 3$. Indeed, we have

$$(f, g, h) \ \subseteq \ I \ \subset \ I : P \ = \ P^2 + (ax + by).$$

So if $\deg(ax+by) \geqslant 4$, then $(f, g, h) \subset P^2$ and by Lemma 2.2, $I \subseteq P^2$ — a contradiction, as $(R/I)_P$ has Hilbert function $(1, 1, 1)$.

If $\deg(ax + by) = 3$, then there are linear forms l_{ij} and field coefficients α, β, γ such that

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & \alpha \\ l_{21} & l_{22} & l_{23} & \beta \\ l_{31} & l_{32} & l_{33} & \gamma \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \\ ax + by \end{pmatrix}.$$

As $(f, g, h) \not\subset (x, y)^2$, one of the coefficients α, β, γ is non-zero; say $\alpha \neq 0$. Setting $a' = l_{11}x + l_{12}y + \alpha a$ and $b' = l_{13}y + \alpha b$, we have $f = a'x + b'y$. As $\alpha \neq 0$ and x, y, a, b form a regular sequence, so do x, y, a', b'. By Lemma 2.11 the ideal $P^3 + (f)$ is unmixed of multiplicity 3 and by Lemma 2.10 it is equal to I. This allows us now to compute the link $(x^3, y^3) : I = \big(x^3, x^2y^2, y^3, (a'x - b'y)xy, x^2a'^2 - xya'b' + y^2b'^2\big)$, which has projective dimension 3. Thus, $\mathrm{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.

It remains the case $\deg(ax+by)=2$. As illustrated in Example 3.7, the cubics f,g,h can be expressed in terms of the quadrics $x^2, xy, y^2, ax+by$ using no more than 12 linear forms l_{ij}. (Note that a,b are linear as well.) Without having determined the unmixed part I in this case, we use the fact $f,g,h \in k[l_{ij}, a, b, x, y]$ to infer that $\operatorname{pd}(R/(f,g,h)) \leqslant 16$.

$\underline{\langle e=1,2; \lambda=1,1\rangle}$ The unmixed part I is the intersection of two height two prime ideals with multiplicities 1 and 2, respectively. Write $I=(u,v)\cap(x,q)$ with linear forms u,v,x and an irreducible quadric q. Note that after subtracting a suitable multiple of x from q, we may assume that q is reduced modulo x without changing the ideal (x,q). As neither the ideal (u,v) nor (x,q) is contained in the other, we know that $\operatorname{ht}(u,v,x,q)=3$ or 4.

If $\operatorname{ht}(u,v,x,q)=3$, then it is easily seen that either x or q belongs to (u,v). (Indeed, as q is reduced modulo x, the condition $q\notin(u,v)$ is tantamount to $q\notin(u,v,x)$. So if in addition $x\notin(u,v)$, then u,v,x,q would form a regular sequence and $\operatorname{ht}(u,v,x,q)=4$.) Thus, (u,v,x,q) is generated by a regular sequence of length three — either by (u,v,q) if $x\in(u,v)$ or by (u,v,x) if $q\in(u,v)$ — and $\operatorname{pd}(R/(u,v,x,q))=3$. Applying Lemma 2.1 to the short exact sequence

$$0 \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(u,v)} \oplus \frac{R}{(x,q)} \longrightarrow \underbrace{\frac{R}{(u,v,x,q)}}_{\text{proj. dim.}=3} \longrightarrow 0$$

yields $\operatorname{pd}(R/I)=2$. As I has height two, R/I is Cohen-Macaulay and therefore $\operatorname{pd}(R/(f,g,h))\leqslant 3$ by Theorem 3.1.

If $\operatorname{ht}(u,v,x,q)=4$, then u,v,x,q form a regular sequence and $I=(ux,uq,vx,vq)$. We compute the link $(ux,vq):I=(x,v)\cap(u,q)$, which has projective dimension 3. Thus, $\operatorname{pd}(R/(f,g,h))\leqslant 4$ by Theorem 3.3.

$\underline{\langle e=1,1; \lambda=1,2\rangle}$ By Proposition 2.13, I admits a primary decomposition either of the form $(u,v)\cap(x,y^2)$ or of the form $(u,v)\cap(x^2,xy,y^2,ax+by)$ with independent linear forms u,v, independent linear forms x,y, and elements $a,b\in R$ such that x,y,a,b form a regular sequence. Since there is no containment among the associated primes (x,y) and (u,v), we have $\operatorname{ht}(x,y,u,v)=3$ or 4.

The case $I = (u, v) \cap (x, y^2)$ is completely analogous to the case $\langle e = 1, 2; \lambda = 1, 1\rangle$ above, with the quadric q replaced by y^2. (The arguments used did not rely on the fact that q was irreducible.)

So let $I = (u, v) \cap (x^2, xy, y^2, ax + by)$. Note that removing any multiples of x or y from a and b does not change the ideal $(x^2, xy, y^2, ax + by)$, as this amounts to the reduction of the term $ax + by$ modulo $(x, y)^2$. Hence we may assume that a, b are reduced modulo (x, y).

Throughout the subsequent arguments we will occasionally use the following simple fact in order to analyze the intersection $(u, v) \cap (x^2, xy, y^2, ax + by)$.

Lemma 3.8. *If K_1, K_2, L are ideals with $K_2 \subseteq L$, then $L \cap (K_1 + K_2) = [L \cap K_1] + K_2$.*

Proof. The inclusion "$\supseteq$" is clear. As for "$\subseteq$", let $l = k_1 + k_2$ be an element in $L \cap (K_1 + K_2)$ with $l \in L$, $k_1 \in K_1$, and $k_2 \in K_2$. By assumption $k_2 \in L$. So $k_1 = l - k_2 \in L \cap K_1$ and $k_1 + k_2 \in [L \cap K_1] + K_2$. $\square$

As argued previously in the case $\langle e = 1; \lambda = 3\rangle$, we have $\deg(ax + by) \leqslant 3$. For if $\deg(ax + by) \geqslant 4$, then the inclusion $(f, g, h) \subseteq I = (u, v) \cap (x^2, xy, y^2, ax + by)$ would imply $(f, g, h) \subset (u, v) \cap (x, y)^2$ and by Lemma 2.2, $I \subseteq (x, y)^2$ — a contradiction.

The case $\deg(ax + by) = 1$ corresponds to $I = (u, v) \cap (x, y^2)$ and was discussed above. So we proceed with $\deg(ax + by) = 2$, that is, a and b are linear forms. In particular, after choosing suitable generators for the ideal (x, y), we may assume that a is reduced modulo b or vice versa: If, say, $a = a' + \beta b$, then $ax + by = a'x + b(\beta x + y)$ and we can relabel a' as a and $\beta x + y$ as y without changing the ideal $(x, y)^2 + (ax + by)$.

- If $\mathrm{ht}(x, y, u, v) = 3$, then we may relabel u as x. We also know that $y \notin (x, v)$. To calculate the intersection $(x, v) \cap (x^2, xy, y^2, ax + by)$, we distinguish between the following two cases:

  i) If $ax + by \in (x, v)$, then by Lemma 3.8

$$I \;=\; \big[(x, v) \cap (x, y)^2\big] + (ax + by) \;=\; (x^2, xy, y^2 v, ax + by).$$

However, $ax + by \in (x, v)$ is equivalent to $by \in (x, v)$. And as $y \notin (x, v)$ and b is reduced modulo x, that gives us $b \in (v)$. Since b is linear, we may rescale so that $b = v$. One now sees that the generator y^2v is redundant and I is generated by the 2×2 minors of a 3×2 matrix:

$$I \;=\; (x^2, xy, ax+by) \;=\; I_2\begin{pmatrix} x & 0 \\ a & -y \\ b & x \end{pmatrix}.$$

As I has height two, R/I is Cohen-Macaulay and $\operatorname{pd}(R/(f,g,h)) \leqslant 3$ by Theorem 3.1.

ii) If $ax+by \notin (x, v)$, then we claim that $I = \big(x^2, xy, y^2v, (ax+by)v\big)$. Clearly the ideal $\big(x^2, xy, y^2v, (ax+by)v\big)$ has height two and it is contained in $I = (x, v) \cap (x^2, xy, y^2, ax+by)$. So, by Lemma 2.10, it suffices to show that it is unmixed of multiplicity 3. This in turn follows from Lemma 2.12 once we verify that $\operatorname{ht}(x, y, av, b) = 4$. As x, y, a, b already form a regular sequence, it suffices to show that $b \notin (x, y, v)$. Given that b is reduced modulo (x, y), this is tantamount to $b \notin (v)$, which follows from the hypothesis $ax+by \notin (x, v)$. So $I = \big(x^2, xy, y^2v, (ax+by)v\big)$. We now claim that

$$(x^2, y^2v) : I \;=\; \big(x^2, xy, y^2v, (ax-by)v\big)\,.$$

To show this, we first observe that $\big(x^2, xy, y^2v, (ax-by)v\big) \subseteq (x^2, y^2v) : I$, which amounts to checking $\big(xy, (ax-by)v\big)\big(xy, (ax+by)v\big) \subseteq (x^2, y^2v)$. As $(x^2, y^2v) : I$ has height two and multiplicity 3, our claim follows from Lemmas 2.12 and 2.10. Lemma 2.12 also asserts that $\operatorname{pd}\big({}^{R}/_{(x^2,y^2v):I}\big) \leqslant 3$, so $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.3.

- Now suppose $\operatorname{ht}(x, y, u, v) = 4$. As above, we distinguish between the following two cases:

i) If $ax+by \in (u, v)$, then $I = \big[(u, v) \cap (x, y)^2\big] + (ax+by)$ by Lemma 3.8. Since a, b are reduced modulo (x, y), no cancellation among the terms ax and by is possible. So $ax+by \in (u, v)$ implies that both $ax, by \in (u, v)$. Since

$x, y \notin (u,v)$ by assumption on the height of (x,y,u,v), this means that $a, b \in (u,v)$. But a, b are linear and linearly independent, so $(a,b) = (u,v)$ and the terms axy and bxy become redundant:

$$I \;=\; (ax^2, bx^2, ay^2, by^2, ax+by).$$

The link $(ax^2, by^2) : I$ is the ideal $\big(ax^2, by^2, x^2y^2, abxy, (ax-by)ab\big)$, which has projective dimension 3. Hence $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.3.

ii) If $ax+by \notin (u,v)$, then $(a,b) \not\subset (u,v)$. Say $a \notin (u,v)$. As a is reduced modulo (x,y), we have $a \notin (x,y,u,v)$ and $\mathrm{ht}(x,y,u,v,a) = 5$.

If $b \notin (x,y,u,v,a)$, then x,y,u,v,a,b are independent linear forms and

$$I \;=\; \big((x,y)^2 + (ax+by)\big)\ (u,v).$$

We compute the link

$$(x^2u, y^2v) : I \;=\; \left(x^2u, y^2v, x^2y^2, xyuv, (ax-by)uv\right),$$

which has projective dimension 3. By Theorem 3.3, $\mathrm{pd}(R/(f,g,h)) \leqslant 4$.

If on the other hand $b \in (x,y,a,u,v)$, then, after reduction modulo (x,y,a), we have $b = \alpha u + \beta v$. Without loss of generality $\alpha \neq 0$ and we may replace u by b. Now x,y,a,b,v are independent linear forms and as above

$$I \;=\; \big((x,y)^2 + (ax+by)\big)\ (b,v).$$

We compute the link

$$(x^2b, y^2v) : I \;=\; \left(x^2b, y^2v, x^2y^2, xybv, (ax-by)bv\right),$$

which has projective dimension 3. By Theorem 3.3, $\mathrm{pd}(R/(f,g,h)) \leqslant 4$.

It remains the case where a, b are quadrics and $\deg(ax+by) = 3$. We first reduce to the situation where we may assume that $ax+by \in (u,v)$. Recall that we cannot have $(f,g,h) \subset (x,y)^2$, for otherwise $I \subset (x,y)^2 \cap (u,v)$ by Lemma 2.2 — a contradiction.

So suppose without loss of generality that f has a non-zero contribution from the cubic term $ax + by$, that is,

$$f \;=\; l_1\, x^2 + l_2\, xy + l_3\, y^2 + \alpha(ax + by)$$

with linear forms l_1, l_2, l_3 and a scalar $0 \neq \alpha \in k$. Setting $a' = l_1\, x + l_2\, y + \alpha a$ and $b' = l_3\, y + \alpha b$, we have $f = a'x + b'y$ where x, y, a', b' form a regular sequence. By Lemma 2.11 the ideal $(x, y)^2 + (f)$ is unmixed of multiplicity 2 and by Lemma 2.10 it is equal to $(x, y)^2 + (ax + by)$. So we may replace $ax + by$ by $f = a'x + b'y$ without changing the ideal I. Note that $f \in I \subset (u, v)$. To ease notation, we relabel a' as a and b' as b and arrive at $I = \big[(u, v) \cap (x, y)^2\big] + (ax + by)$ with $ax + by \in (u, v)$.

As before, we consider the following two cases:

- If $\operatorname{ht}(x, y, u, v) = 3$, relabel u as x and note that $y \notin (x, v)$. Then $ax + by \in (x, v)$ implies $b \in (x, v)$, say $b = b_1 x + b_2 v$ with linear forms b_1, b_2. Note that $b_2 \notin (x, y)$. We have $ax + by = (a + b_1 y)x + b_2 vy$. Relabeling $a + b_1 y$ as a and b_2 as b', we write $I = (x^2, xy, y^2 v, ax + b'yv)$. Observe that $a \notin (x, y, b')$, for that would imply $(x, y, a, b) \subset (x, y, b')$. So $\operatorname{ht}(x, y, a, b') = 4$. We now claim that

  $$(x^2, y^2 v) : I \;=\; (x^2, xy, y^2 v, ax - b'vy).$$

  To see this, we observe that $(x^2, xy, y^2 v, ax - b'vy) \subseteq (x^2, y^2 v) : I$, which amounts to checking $(xy, ax - b'vy)(xy, ax + b'vy) \subseteq (x^2, y^2 v)$. As $(x^2, y^2 v) : I$ has height two and multiplicity 3, our claim follows from Lemmas 2.12 and 2.10. Lemma 2.12 also asserts that $\operatorname{pd}\big({}^{R}/_{(x^2, y^2 v):I}\big) \leqslant 3$, so $\operatorname{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.

- If $\operatorname{ht}(x, y, u, v) = 4$, then $I = (x^2 u, x^2 v, xyu, xyv, y^2 u, y^2 v, ax + by)$. (Note that in this case I is generated by cubic forms and does not contain any quadrics.) As x, y, u, v form a regular sequence, $ax + by \in (u, v)$ implies $a \in (y, u, v)$ and $b \in (x, u, v)$. A priori we would need six linear coefficients in order to write a and b in terms of x, y, u, v. However, after combining the coefficients of xy in $ax + by$, this number can be reduced to 5. So, the (cubic) generators of I can be expressed entirely in terms of 9 linear forms. Hence $\operatorname{pd}(R/(f, g, h)) \leqslant 9$.

$\langle e = 1, 1, 1; \lambda = 1, 1, 1\rangle$ I is the intersection of three prime ideals of height two and multiplicity one. Write $I = (x, y) \cap (u, v) \cap (s, t)$ with linear forms x, y, u, v, s, t and note that $3 \leqslant \operatorname{ht}(x, y, u, v, s, t) \leqslant 6$. We consider each case by giving an explicit description of the ideal I, up to a linear change of coordinates:

- If $\operatorname{ht}(x, y, u, v, s, t) = 3$, then $I = (x, y) \cap (x, u) \cap (y, u) = (xy, xu, yu)$. R/I is Cohen-Macaulay and $\operatorname{pd}(R/(f, g, h)) \leqslant 3$ by Theorem 3.1.
- If $\operatorname{ht}(x, y, u, v, s, t) = 4$, then either $I = (x, y) \cap (y, u) \cap (u, v) = (xu, yu, yv)$ or $I = (x, y) \cap (x, u) \cap (x, v) = (x, yuv)$. In both cases R/I is Cohen-Macaulay and $\operatorname{pd}(R/(f, g, h)) \leqslant 3$.
- If $\operatorname{ht}(x, y, u, v, s, t) = 5$, then $I = (x, y) \cap (u, v) \cap (v, s) = (xv, yv, xus, yus)$. The link $(xv, yus) : I$ is the ideal (xy, xv, yus, vus), which has projective dimension 3. Hence $\operatorname{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.
- If $\operatorname{ht}(x, y, u, v, s, t) = 6$, then $I = (xus, xut, xvs, xvt, yus, yut, yvs, yvt)$. The link $(xus, yvt) : I$ is the ideal $(xus, yvt, xyuv, xyst, uvst)$, which has projective dimension 3. Hence $\operatorname{pd}(R/(f, g, h)) \leqslant 4$ by Theorem 3.3.

3.2.4 The Case of the Unmixed Part Containing a Quadric

With the knowledge gained so far about the projective dimension of three cubics of multiplicity $\leqslant 3$, we are able to prove the following theorem which may be viewed as an extension of Corollary 3.2.

Theorem 3.9. *Let $R = k[X_1, \ldots, X_n]$ and let $J \subset R$ be an ideal which is generated by three cubic forms. If the unmixed part of J contains a quadric form, then* $\operatorname{pd}(R/J) \leqslant 4$.

Proof. As shown in the introduction of Section 3.2, we may assume that $\operatorname{ht}(J) = 2$, for otherwise $\operatorname{pd}(R/J) \leqslant 4$. But for a height two ideal J we have already shown without further assumptions that $\operatorname{pd}(R/J) \leqslant 3$ if $e(R/J) = 1$ and $\operatorname{pd}(R/J) \leqslant 4$ if $e(R/J) = 2$. Thus, we may further assume that $e(R/J) \geqslant 3$.

Now we make use of the assumption that the unmixed part of J contains a quadric. Denote by I the unmixed part of J and let $q \in I$ be a quadric. As J is generated by cubics and $\operatorname{ht}(J) = 2$, we can choose a cubic form $f \in J \subseteq I$ such that q and f form a regular sequence. Note that this imposes an upper bound on the multiplicity of R/I. Namely, $e(R/I) \leqslant e(R/(q,f)) = 6$ and, by Lemma 2.10, equality holds if and only if $I = (q,f)$, in which case $\operatorname{pd}(R/J) \leqslant 3$ by Theorem 3.1. So there remain the cases $3 \leqslant e(R/I) \leqslant 5$.

Consider the link $(q,f) : I$ which has multiplicity $6 - e(R/I)$. If $e(R/I) = 5$, then $(q,f) : I$ is a height two, unmixed ideal of multiplicity 1, so it is generated by two linear forms and $\operatorname{pd}(R/J) \leqslant 3$ by Theorem 3.3. If $e(R/I) = 4$, then $(q,f) : I$ is a height two, unmixed ideal of multiplicity 2. Such ideals were classified in Proposition 2.13 whereby $\operatorname{pd}\big({}^{R}/_{(q,f):I}\big) \leqslant 3$. Thus, $\operatorname{pd}(R/J) \leqslant 4$ by Theorem 3.3.

It remains the case $e(R/I) = 3$. Note that now the link $(q,f) : I$ has multiplicity 3 as well. If I is of type $\langle e = 3; \lambda = 1\rangle$, $\langle e = 1,2; \lambda = 1,1\rangle$, or $\langle e = 1,1,1; \lambda = 1,1,1\rangle$, then we have already shown (without the assumption that I contains a quadric) that $\operatorname{pd}(R/(f,g,h)) \leqslant 4$. As for ideals of type $\langle e = 1,1; \lambda = 1,2\rangle$, the only instance where we obtained a bound for $\operatorname{pd}(R/(f,g,h))$ greater than 4 was that of the unmixed part generated entirely by cubics — see page 55. So we are left with the remaining case where I is of type $\langle e = 1; \lambda = 3\rangle$, that is, I is primary to (x,y) with independent linear forms x, y. Note that $(x,y)^3 \subsetneq I$.

First we assume that $q \notin (x,y)^2$. Say $q = cx + dy$ with $(c,d) \not\subset (x,y)$, that is, $\operatorname{ht}(x,y,c,d) \geqslant 3$. If $\operatorname{ht}(x,y,c,d) = 4$, then $(x,y)^3 + (cx+dy)$ is unmixed by Lemma 2.11 and $I = (x,y)^3 + (cx+dy)$ by Lemma 2.10, in which case the link $(x^3,y^3) : I = \big(x^3, x^2y^2, y^3, (cx-dy)xy, x^2c^2 - xycd + y^2d^2\big)$ has projective dimension 3 and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.3. If on the other hand $\operatorname{ht}(x,y,c,d) = 3$, then we may choose q to be of the form $cx + \alpha xy + y^2$ with some field coefficient α.

Indeed, suppose $\operatorname{ht}(x,y,c,d) = 3$ where $c \notin (x,y)$ while $d = \alpha x + \beta y + \gamma c$ with field coefficients α, β, γ. Note that $\beta \neq \alpha\gamma$. (If $\beta = \alpha\gamma$, then $cx + dy = (c + \alpha y)(x + \gamma y)$. That is, $c + \alpha y$ would be a zerodivisor on I, contradicting $c \notin (x,y)$.) We have

$cx + dy = c(x+\gamma y) + (\alpha x + \beta y)y$. Relabeling $x+\gamma y$ as x and rescaling y yields $q = cx + \alpha xy + y^2$. Clearly, if $\alpha \neq 0$, then we may also rescale x and write q simply as $cx + xy + y^2$. (We shall see in retrospect that the value of α has no bearing on the form of the unmixed part I, as in either case xy is contained in I.)

We are now able to compute the link $(q, y^3) : I$. First we compute the colon $(q, y^3) : \big((q) + (x,y)^3\big)$ by hand, using the fact that x, y, c are independent linear forms:

$$(q, y^3) : \big((q) + (x,y)^3\big) \;=\; \begin{cases} (c^2 - y^2, cy + y^2, cx + xy + y^2) & \text{if } \alpha = 1, \\ (c^2, cy, cx + y^2) & \text{if } \alpha = 0. \end{cases} \tag{3.7}$$

In either case the ideal $(q, y^3) : \big((q) + (x,y)^3\big)$ has multiplicity 3 and clearly it contains the ideal $(q, y^3) : I$, since $(q) + (x,y)^3 \subseteq I$. By Lemma 2.10 we have equality:

$$(q, y^3) : I \;=\; (q, y^3) : \big((q) + (x,y)^3\big)\,.$$

It is easily seen that both ideals in (3.7) have Cohen-Macaulay quotients, as they are generated by the 2×2 minors of a 3×2 matrix. Thus, $\operatorname{pd}(R/I) = \operatorname{pd}\big({}^{R}\!/_{(q,y^3):I}\big) = 2$ and $\operatorname{pd}(R/(f,g,h)) \leqslant 3$.

(Note: Linking the ideal $(q, y^3) : I$ back to $I = (q, y^3) : \big((q, y^3) : I\big)$ reveals that $I = (x^2, xy, cx + y^2)$ regardless of the value of α.)

It remains the case where $q \in (x,y)^2$. Since we may assume without loss of generality that the ground field k is algebraically closed[1], we can factor the quadric q as $l\,l'$ where $l, l' \in (x,y) \setminus I$ are two (not necessarily independent) linear forms. Now consider the following chain of (x,y)-primary ideals:

$$I \;\subsetneq\; I : (x,y) \;\subseteq\; I : l \;\subseteq\; (x,y).$$

We have $e(R/I) = 3$ and $e(R/(x,y)) = 1$. Furthermore, $e\big({}^{R}\!/_{I:(x,y)}\big) = 1$ if and only if $I : (x,y) = (x,y)$ and $I = (x,y)^2$. So we may assume $e\big({}^{R}\!/_{I:(x,y)}\big) = 2$. That forces $e\big({}^{R}\!/_{I:l}\big) = 2$ or 1. In what follows we show what each of these multiplicity values entails for the structure of $I : l$, and subsequently for that of I. As it turns out, the results are the same in either case.

[1]See remark on page 6 in Section 1.1.

First suppose $I : l$ has multiplicity 1. Then $I : l = (x, y)$ and after a linear change of coordinates we may relabel l as x. So $x^2, xy \in I$ and by Lemma 2.15 we have $I = (x^2, xy, y^3, cx + dy^2)$ with elements c and d such that $\operatorname{ht}(x, y, c, d) > 3$. (We are assuming $I \neq (x, y)^2$.) As the link $(x^2, y^3) : I = (x^2, xy, y^3, cx - dy^2)$ has projective dimension 3, we have $\operatorname{pd}(R/(f, g, h)) \leqslant 4$.

Now suppose $I : l$ has multiplicity 2. (In particular, $I : l = I : (x, y)$.) By Proposition 2.13 we have $I : l = (x, y)^2 + (ax + by)$ with elements a, b such that $\operatorname{ht}(x, y, a, b) > 3$. As $q = l\,l' \in I$ and $l' \in I : l$, the term $ax + by$ must be linear. Relabeling $ax + by$ as x, we have $I : (x, y) = (x, y^2)$ and again $I = (x^2, xy, y^3, cx + dy^2)$ by Lemma 2.15 and $\operatorname{pd}(R/(f, g, h)) \leqslant 4$ as above.

(Note: The scenario $e\left({}^{R}\!/\!_{I:l}\right) = 1$ corresponds to the choice of $q = x^2$ and $l = l' = x$, while $e\left({}^{R}\!/\!_{I:l}\right) = 2$ corresponds to $q = xy$, $l = y$, and $l' = x$.) □

We point out that Theorem 3.9 cannot be further improved. Its conclusion cannot be strengthened, as illustrated in Examples 3.6 and 3.7, and its hypothesis cannot be weakened: It is possible for R/J to have projective dimension greater than 4 if the unmixed part of J is generated in degrees 3 and higher. In Section 3.3 we give an example with $\operatorname{pd}(R/J) = 5$ where J^{unm} is generated by five cubic forms.

3.2.5 Multiplicity Four

To cope with the case of multiplicity 4, we prove Proposition 3.11 which supplies a bound for $\operatorname{pd}(R/(f, g, h))$ whenever (f, g, h) has multiplicity $\geqslant 2$ along a codimension two linear subspace. For that purpose we shall first demonstrate with the following lemma that under certain circumstances (the ideal generated by) three quadrics can be expressed in terms of a fixed number of linear forms.

Lemma 3.10. *Three quadric forms which minimally generate an ideal of height at most 2 can be expressed entirely in terms of 8 linear forms, unless two of the quadrics generate an ideal of height one, that is, they have a common linear factor.*

Proof. Let q_1, q_2, q_3 be three quadrics. The statement is evident if $\operatorname{ht}(q_1, q_2, q_3) = 1$. In that case $q_i = l_i x$ with $i = 1, 2, 3$ and linear forms l_i, x. That is, $q_1, q_2, q_3 \in k[l_1, l_2, l_3, x]$. If $\operatorname{ht}(q_1, q_2, q_3) = 2$, then the arguments in Section 2.2 which led to Lemma 2.9, mutatis mutandis, show that the ideal (q_1, q_2, q_3) has multiplicity at most 3. We pass to the unmixed part of (q_1, q_2, q_3) and consider each case separately:

Let I denote the unmixed part of the ideal (q_1, q_2, q_3) and note that $\operatorname{ht}(I) = 2$. If $e(R/I) = 1$, then $I = (x, y)$ is generated by two independent linear forms x, y and $q_i = l_{i1}x + l_{i2}y$ with $i = 1, 2, 3$ and linear forms l_{i1}, l_{i2}. So q_1, q_2, q_3 can be expressed in terms of 8 linear forms l_{i1}, l_{i2}, x, y.

If $e(R/I) = 2$, then, by Proposition 2.13, I is one of the following ideals:

(i) $I = (x, q)$ with a linear form x and an irreducible quadric q. Then $q_i = l_i x + \alpha_i q$ with linear forms l_i and field coefficients α_i for $i = 1, 2, 3$. As $\operatorname{ht}(q_1, q_2, q_3) = 2$, the coefficients α_i must not be all zero; say $\alpha_3 \neq 0$. Replacing q_1 by $q_1 - \frac{\alpha_1}{\alpha_3} q_3 = (l_1 - \frac{\alpha_1}{\alpha_3} l_3)x$ and q_2 by $q_2 - \frac{\alpha_2}{\alpha_3} q_3 = (l_2 - \frac{\alpha_2}{\alpha_3} l_3)x$, they both become divisible by the linear form x and we are done.

(ii) $I = (x, yv)$ with independent linear forms x, y, v. Then $q_i = l_i x + \alpha_i yv$ with linear forms l_i and field coefficients α_i for $i = 1, 2, 3$. So $q_1, q_2, q_3 \in k[l_1, l_2, l_3, x, y, v]$.

(iii) $I = (xu, xv, yu, yv)$ with independent linear forms x, y, u, v. Clearly we have $q_1, q_2, q_3 \in k[x, y, u, v]$.

(iv) $I = (x, y)^2 + (ax + by)$ with independent linear forms x, y and elements $a, b \in \mathfrak{m}$ such that x, y, a, b form a regular sequence. As I is the unmixed part of an ideal generated by quadrics, we must have $\deg(ax + by) = 2$, for otherwise $I = (x, y)^2$ by Lemma 2.2. So, a and b are linear and $q_1, q_2, q_3 \in k[a, b, x, y]$.

(iv°) $I = (x, y^2)$ with independent linear forms x, y. In analogy to part (ii) above, $q_1, q_2, q_3 \in k[l_1, l_2, l_3, x, y]$.

It remains the case $e(R/I) = 3$. We review all five possible choices for I. (See page 24 in Section 2.3 for explanation of the subsequent notation.)

$\underline{\langle e = 1; \lambda = 3 \rangle}$ I is primary to (x, y), where x, y are independent linear forms, and

$\lambda\big((R/I)_{(x,y)}\big) = 3$. Either $I = (x,y)^2$ or I is generated by $(x,y)^3$ plus additional terms of the form $c_j x + d_j y$ with $(c_j, d_j) \not\subset (x,y)^2$. In the former case we are done, as $q_1, q_2, q_3 \in k[x,y]$. In the latter case we first rule out the possibility that one of the terms $c_j x + d_j y$ may be linear: if so, then $I = (x, y^3)$ after a linear change of coordinates and thus $(q_1, q_2, q_3) \subset (x)$ — a contradiction, since $\operatorname{ht}(q_1, q_2, q_3) = 2$.

So now we have $(q_1, q_2, q_3) \subseteq (c_j x + d_j y)$ with $\deg(c_j x + d_j y) \geqslant 2$. Write $q_i = \sum_j \alpha_{ij}(c_j x + d_j y)$ with field coefficients α_{ij} where $\alpha_{ij} = 0$ whenever $\deg(c_j x + d_j y) > 2$. That is, $l_{i1} = \sum_j \alpha_{ij} c_j$ and $l_{i2} = \sum_j \alpha_{ij} d_j$ are linear and q_1, q_2, q_3 can be expressed in terms of 8 linear forms l_{i1}, l_{i2}, x, y.

$\underline{\langle e = 3; \lambda = 1\rangle}$ I is the ideal generated by the 2×2 minors of a 3×2 matrix of indeterminates — see Section 2.4.1. That is, I is generated by three quadrics in at most six variables, and therefore the same holds for (q_1, q_2, q_3).

Finally, in the remaining three cases $\langle e = 1,1; \lambda = 1,2\rangle$, $\langle e = 1,2; \lambda = 1,1\rangle$, and $\langle e = 1,1,1; \lambda = 1,1,1\rangle$, I is contained in an ideal of type $\langle e = 1; \lambda = 1\rangle$, that is, $I \subset (x,y)$ with linear forms x, y. As argued in the case of multiplicity one, the quadrics q_1, q_2, q_3 can be expressed in terms of 8 linear forms. □

Proposition 3.11. *Let f, g, h be three cubic forms which minimally generate an ideal of height two. Suppose that (f,g,h) has a component primary to an ideal $P = (x,y)$ with independent linear forms x, y and $\lambda\Big(\big({}^{R}\!/_{(f,g,h)}\big)_P\Big) \geqslant 2$. Then $\operatorname{pd}(R/(f,g,h)) \leqslant 36$.*

(In our notation, the hypothesis of the proposition simply states that if (f,g,h) is of type $\langle e = a_1, \ldots, a_m; \lambda = b_1, \ldots, b_m\rangle$, then $a_i = 1$ and $b_i \geqslant 2$ for some i.)

Proof. Let Q denote the P-primary component of (f,g,h), that is, $(f,g,h) \subseteq Q \subsetneq P$ and $(f,g,h)_P = Q_P \subsetneq P_P$. We have $e(R/Q) = \lambda(R_P/Q_P) \geqslant 2$. If $Q \subseteq P^2$, then the cubics f, g, h can be expressed in terms of the quadrics x^2, xy, y^2 using no more than 9 linear forms l_i, in which case $f, g, h \in k[x, y, l_i]$ and $\operatorname{pd}(R/(f,g,h)) \leqslant 11$. So we may assume that Q contains terms of the form $cx + dy$ where $(c,d) \not\subset P$. Consequently the Hilbert function of $(R/Q)_P$ is given by $(\underbrace{1,1,1,\ldots,1}_{e(R/Q)\text{ times}})$.

(We caution that in addition to $P^{e(R/Q)}$ and the above mentioned terms of the form $cx+dy$ with $(c,d) \not\subset P$, the ideal Q may contain other terms as minimal generators — see Example 2.14.)

Now consider the ideal $I := Q : P^{e(R/Q)-2}$ whose Hilbert function, locally at P, is given by $(1,1)$. That is, I is a P-primary ideal of multiplicity 2. By parts (iv) and (iv°) of Proposition 2.13, $I = P^2 + (ax+by)$ with elements a, b such that $\mathrm{ht}(x,y,a,b) > 3$. (Here the term $ax+by$ need not be the same as the term $cx+dy$ above.) In other words, either x, y, a, b form a regular sequence or (a,b) is the unit ideal, in which case we may take I to be (x, y^2). Note that $(f,g,h) \subseteq Q \subseteq P^2 + (ax+by)$. In what follows, we exploit this inclusion to put f, g, h inside a subring of R generated by a bounded number of linear forms (or by a regular sequence), which will in turn give a bound for $\mathrm{pd}(R/(f,g,h))$.

If $\deg(ax+by) = 4$, then $(f,g,h) \subseteq P^2$ and $\mathrm{pd}(R/(f,g,h)) \leqslant 11$ as shown above. (Strictly speaking, this case is ruled out by our above assumption that $Q \not\subset P^2$.) If $\deg(ax+by) = 3$, then, without loss of generality, we may take $h = ax+by$ and have $f, g \in P^2$. Indeed, as $(f,g,h) \subseteq P^2 + (ax+by)$, there are nine linear forms l_{ij} and field coefficients α, β, γ such that

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & \alpha \\ l_{21} & l_{22} & l_{23} & \beta \\ l_{31} & l_{32} & l_{33} & \gamma \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \\ ax+by \end{pmatrix}.$$

If $\alpha = \beta = \gamma = 0$, then $(f,g,h) \subseteq P^2$ and we are done; so we may assume that $\gamma \neq 0$. Replacing f by $f - \frac{\alpha}{\gamma}h$ and g by $g - \frac{\beta}{\gamma}h$, we have $f, g \in P^2$. And relabeling $(l_{31}x + l_{32}y + \gamma a)$ as a and $(l_{33}y + \gamma b)$ as b, we can write $h = ax+by$ where x, y, a, b still form a regular sequence. Now we set $L := (l_{11}, l_{12}, l_{13}, l_{21}, l_{22}, l_{23})$ and consider the following two cases:

If a and b share a common factor c modulo $L+P$, then $\mathrm{pd}(R/(f,g,h)) \leqslant 27$. Indeed, if $a \equiv a'c$ and $b \equiv b'c$ modulo $L+P$ with some linear forms a', b', c, then $a - a'c$ can be written in terms of $x, y, l_{11}, \ldots, l_{23}$ using eight linear forms $u_1, \ldots, u_8$, and the same holds for $b - b'c$ with eight linear forms $v_1, \ldots, v_8$. Thus, the cubics f, g, h are in the subring $k[x, y, l_{11}, \ldots, l_{23}, c, a', u_1, \ldots, u_8, b', v_1, \ldots, v_8]$ and $\operatorname{pd}(R/(f, g, h)) \leqslant 27$.

If on the other hand a, b do not have a common factor modulo $L + P$, then they form a regular sequence modulo $L + P$. That is, the generators of $L + P$ along with a, b form a regular sequence of length at most 10 and $\operatorname{pd}(R/(f, g, h)) \leqslant 10$.

If $\deg(ax+by) = 2$, then the cubics f, g, h can be expressed in terms of the quadrics $x^2, xy, y^2, ax + by$ using no more than 12 linear forms l_{ij}. So $f, g, h \in k[x, y, a, b, l_{ij}]$ and $\operatorname{pd}(R/(f, g, h)) \leqslant 16$. It remains the case where I is of the form (x, y^2). Here we have three linear forms l_1, l_2, l_3 and three quadrics q_1, q_2, q_3 such that

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \\ q_3 & l_3 \end{pmatrix} \begin{pmatrix} x \\ y^2 \end{pmatrix}.$$

If $\operatorname{ht}(q_1, q_2, q_3) \leqslant 2$, then we can apply Lemma 3.10. Either the quadrics q_1, q_2, q_3 can be expressed in terms of 8 linear forms, or two of the quadrics share a common factor, say $q_1 = uz$ and $q_2 = vz$ with linear forms u, v, z. In the former case we have $\operatorname{pd}(R/(f, g, h)) \leqslant 13$. Namely, f, g, h are in the subring generated by x, y, l_1, l_2, l_3 and the 8 linear forms used to express q_1, q_2, q_3.

In the latter case we are left with eight linear forms $x, y, l_1, l_2, l_3, u, v, z$ and one quadric q_3. If q_3 is in the ideal generated by these eight linear forms, then it can be expressed in terms of those using another set of eight linear forms. So f, g, h are in a subring generated by at most 16 linear forms and $\operatorname{pd}(R/(f, g, h)) \leqslant 16$. And if $q_3 \notin (x, y, l_1, l_2, l_3, u, v, z)$, then q_3 is a non-zerodivisor modulo this ideal, that is, the generators of $(x, y, l_1, l_2, l_3, u, v, z)$ together with q_3 form a regular sequence of length at most 9 and $\operatorname{pd}(R/(f, g, h)) \leqslant 9$.

Lastly, we need to consider the case $\operatorname{ht}(q_1, q_2, q_3) = 3$ where q_1, q_2, q_3 form a regular sequence. If they also do so modulo the ideal (x, y, l_1, l_2, l_3), then we have $\operatorname{pd}(R/(f, g, h)) \leqslant 8$, as the generators of (x, y, l_1, l_2, l_3) along with q_1, q_2, q_3 form a regular sequence of length at most 8. So we may assume that the images of q_1, q_2, q_3 in $R/(x, y, l_1, l_2, l_3)$ generate an ideal of height at most 2. Let $\bar{q}_i$ be the reduction of q_i

modulo (x, y, l_1, l_2, l_3). Note that since $q_i - \bar{q_i} \in (x, y, l_1, l_2, l_3)$, there is a set of 5 linear forms $w_{i1}, \dots, w_{i5}$ such that $q_i - \bar{q_i} \in k[x, y, l_1, l_2, l_3, w_{i1}, \dots, w_{i5}]$.

By Lemma 3.10 either the quadrics $\bar{q_1}, \bar{q_2}, \bar{q_3}$ can be expressed in terms of 8 linear forms, or two of them share a common factor, say $\bar{q_1} = uz$ and $\bar{q_2} = vz$ with linear forms u, v, z. In the former case we can place f, g, h in a subring generated by 28 linear forms: 8 linear forms used to express $\bar{q_1}, \bar{q_2}, \bar{q_3}$, along with x, y, l_1, l_2, l_3 and w_{ij} with $i = 1, 2, 3$ and $j = 1 \dots 5$. Thus, $\operatorname{pd}(R/(f, g, h)) \leqslant 28$.

In the latter case we have $q_1, q_2 \in k[x, y, l_1, l_2, l_3, u, v, z, w_{1j}, w_{2j}]$ with $j = 1 \dots 5$. Consequently, f and g are contained in this subring as well. To obtain h, we need to adjoin q_3. If q_3 is not in the ideal $(x, y, l_1, l_2, l_3, u, v, z, w_{1j}, w_{2j})$, then the generators of this ideals along with q_3 form a regular sequence of length at most 19 and $\operatorname{pd}(R/(f, g, h)) \leqslant 19$. And if q_3 is in the ideal generated by these 18 linear forms, then it can be expressed in terms of those using another set of 18 linear forms. Thus, $\operatorname{pd}(R/(f, g, h)) \leqslant 36$. □

Armed with Theorem 3.9 and Proposition 3.11, we are now able to bound the projective dimension of $R/(f, g, h)$ by 36 in the case of multiplicity 4. By the associativity formula (2.3) there are eleven possible types for the unmixed part I, namely:

$$\langle e = 4; \lambda = 1 \rangle, \qquad \langle e = 1; \lambda = 4 \rangle,$$
$$\langle e = 1, 3; \lambda = 1, 1 \rangle, \qquad \langle e = 1, 1; \lambda = 1, 3 \rangle,$$
$$\langle e = 2, 2; \lambda = 1, 1 \rangle, \qquad \langle e = 1, 1; \lambda = 2, 2 \rangle,$$
$$\langle e = 1, 1, 2; \lambda = 1, 1, 1 \rangle, \qquad \langle e = 1, 1, 1; \lambda = 1, 1, 2 \rangle,$$
$$\langle e = 2; \lambda = 2 \rangle, \qquad \langle e = 1, 2; \lambda = 2, 1 \rangle,$$
$$\langle e = 1, 1, 1, 1; \lambda = 1, 1, 1, 1 \rangle.$$

By virtue of Proposition 3.11 we may dismiss five of these. Namely, we know that $\operatorname{pd}(R/(f, g, h)) \leqslant 36$ whenever the length of R/I is at least two locally at an associated prime ideal of multiplicity 1. There are five such cases which are listed above in the right column. In what follows we consider the remaining six cases.

$\underline{\langle e=4; \lambda=1\rangle}$ If I contains a quadric form, then $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9. So suppose I does not contain any quadrics; in particular, I is non-degenerate and it is not a complete intersection. By part (c) of Theorem 2.18, I is the defining ideal of a generic projection of the Veronese surface $V_5 \subset \mathbb{P}^5$ onto $\mathbb{P}^4$ and it is generated by seven cubic forms (in 5 variables). As f, g, h are linear combinations (with field coefficients) of those cubics, we have $\operatorname{pd}(R/(f,g,h)) \leqslant 5$.

$\underline{\langle e=1,3; \lambda=1,1\rangle}$ I is the intersection $(x,y) \cap P$ of two prime ideals where x, y are linear forms and P is a (height two) ideal of type $\langle e=3; \lambda=1\rangle$. If P contains a linear form l, then I contains a quadric — such as xl or yl — and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9.

If on the other hand P is non-degenerate, then, as outlined in Section 2.4.1, P is the ideal of 2×2 minors of a 3×2 matrix of indeterminates, that is, P is generated by three quadrics in at most six variables. As $(f,g,h) \subseteq I \subset P$, the three cubics f, g, h can be expressed in terms of those quadrics using no more than nine linear coefficients. Thus, $\operatorname{pd}(R/(f,g,h)) \leqslant 15$.

$\underline{\langle e=2,2; \lambda=1,1\rangle}$ I is the intersection $(l_1, q_1) \cap (l_2, q_2)$ of two prime ideals where l_1, l_2 are linear forms and q_1, q_2 are irreducible quadrics. As the quadric $l_1 l_2$ belongs to I, we have $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9.

$\underline{\langle e=1,1,2; \lambda=1,1,1\rangle}$ I is the intersection $(x,y) \cap (u,v) \cap (l,q)$ of three prime ideals where q is an irreducible quadric and x, y, u, v, l are (not necessarily independent) linear forms. If $\operatorname{ht}(x,y,u,v) = 3$, then, without loss of generality, we may replace u by x and write $I = (x, yv) \cap (l, q)$. In this case I contains the quadric xl and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9.

If on the other hand $\operatorname{ht}(x,y,u,v) = 4$, then $I \subset (xu, xv, yu, yv)$ and the cubics f, g, h can be expressed in terms of the quadrics xu, xv, yu, yv using no more than 12 linear forms. Thus, $\operatorname{pd}(R/(f,g,h)) \leqslant 16$.

$\underline{\langle e=2; \lambda=2\rangle}$ I is primary to a prime ideal $P = (l,q)$ with a linear form l and an irreducible quadric q such that $\lambda(R_P/I_P) = 2$. Thus, locally at P, we must have $P_P^2 \subset I_P$. But primary ideals are contracted ideals in the sense that $I = IR_P \cap R$.

Hence $P^2 \subset I$ globally. So I contains the quadric l^2 and we have $\text{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9.

$\underline{\langle e = 1,1,1,1; \lambda = 1,1,1,1\rangle}$ I is the intersection of four height two prime ideals, each of which is generated by two linear forms. So the generators of I are expressed entirely in terms of at most eight (not necessarily independent) linear forms. If I contains a quadric, then $\text{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.9. And if I is generated in degrees 3 and higher, then the cubics f, g, h are linear combinations (with field coefficients) of the cubic generators of I, in which case $\text{pd}(R/(f,g,h)) \leqslant 8$.

3.2.6 Multiplicity Five

The following theorem was inspired by the corresponding statement of Theorem 3.9.

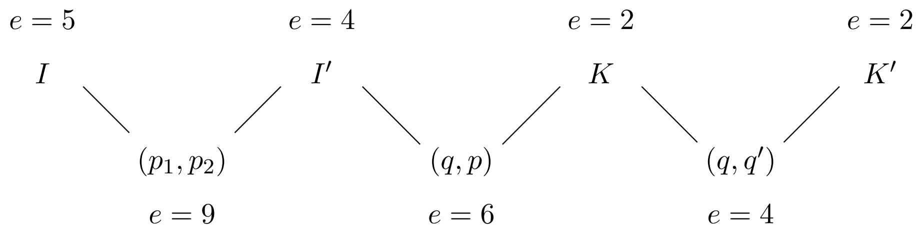
Theorem 3.12. *Let $R = k[X_1, \dots, X_n]$ and let $J \subset R$ be an ideal which is generated by three cubic forms with $\text{ht}(J) = 2$ and $e(R/J) = 3$, 4, or 5. Denote by I the unmixed part of J and let $I' = (p_1, p_2) : I$ be an ideal which is linked to I by two cubic forms $p_1, p_2 \in I$. If I' contains a quadric form, then $\text{pd}(R/J) \leqslant 4$.*

Proof. Let $q \in I'$ be a quadric form. As I' contains cubics which generate an ideal of height two (such as the generators of J or p_1 and p_2), we can choose a cubic $p \in I'$ such that q and p form a regular sequence. Otherwise every cubic in I' is contained in some associated prime of $R/(q)$ and therefore, by prime avoidance, all cubics in I' are contained in an associated prime $P \in \text{Ass}(R/(q))$ — a contradiction, as $\text{ht}(P) = 1$.

If $e(R/I) = 3$, then $e(R/I') = 9 - 3 = 6$ and $I' = (q, p)$ by Lemma 2.10. In this case R/I' (and R/I) are Cohen-Macaulay and $\text{pd}(R/J) \leqslant 3$ by Theorem 3.3. If $e(R/I) = 4$, then $e(R/I') = 9 - 4 = 5$ and we consider a further link $K = (q, p) : I'$. As $e(R/K) = 6 - 5 = 1$, K is generated by two independent linear forms. Thus R/K, R/I', and R/I are Cohen-Macaulay and again $\text{pd}(R/J) \leqslant 3$.

It remains the case $e(R/I) = 5$ with $e(R/I') = 9 - 5 = 4$ and $e(R/K) = 6 - 4 = 2$. It follows from Proposition 2.13 that K contains a second quadric q' such that q and

q' form a regular sequence. So we consider yet another link $K' = (q, q') : K$ with $e(R/K') = 4 - 2 = 2$.



Now we have $\mathrm{pd}(R/K') \leqslant 3$ by Proposition 2.13, $\mathrm{pd}(R/I') = \mathrm{pd}(R/K')$ by Lemma 2.6, and $\mathrm{pd}(R/J) \leqslant \mathrm{pd}(R/I') + 1$ by Theorem 3.3. Thus, $\mathrm{pd}(R/J) \leqslant 4$ as claimed. $\square$

Before proceeding with the case of multiplicity 5, we single out the following argument which we will refer to multiple times in this section as well as in the next. Note that the statement does not contain any assumptions on the multiplicity of the ideal.

Remark 3.13. Let Q be an ideal primary to (x, y) with independent linear forms x, y and let $p_1, \ldots, p_k$ be cubic forms in Q. Suppose $Q \subseteq (x, y)^2 + (ax + by)$ with elements $a, b \in \mathfrak{m}$ such that x, y, a, b form a regular sequence. (In particular, $\deg(ax + by) \geqslant 2$.) Then either Q is of the form $(x, y)^{e(R/Q)} + (a'x + b'y)$ with quadrics a', b' such that x, y, a', b' form a regular sequence, in which case $\mathrm{pd}(R/Q) \leqslant 3$, or the cubics $p_1, \ldots, p_k$ can be expressed entirely in terms of $4(k + 1)$ linear forms.

Proof. The proof of the claim is mainly based on the inclusion

$$(p_1, \ldots, p_k) \;\subseteq\; Q \;\subseteq\; (x, y)^2 + (ax + by).$$

The only obstacle occurs when $\deg(ax + by) = 3$, in which case a and b are quadrics and may involve an arbitrary large number of linear forms.

If $\deg(ax + by) = 3$, then we first consider the case where one of the p_i has a non-zero contribution from the term $ax + by$, that is, if we write

$$p_i \;=\; l_{i1}\, x^2 + l_{i2}\, xy + l_{i3}\, y^2 + \alpha_i\, (ax + by), \qquad i = 1, \ldots, k \tag{3.8}$$

with linear forms l_{ij} and scalars $\alpha_i \in k$, then α_i is non-zero for some i. Say $\alpha_1 \neq 0$. In this case we write p_1 as

$$\begin{aligned} p_1 &= l_{11}\, x^2 + l_{12}\, xy + l_{13}\, y^2 + \alpha_1\, (ax + by) \\ &= \underbrace{(\alpha_1 a + l_{11} x)}_{a'}\, x + \underbrace{(\alpha_1 b + l_{12} x + l_{13} y)}_{b'}\, y, \end{aligned} \tag{3.9}$$

and we note that since the elements x, y, a, b form a regular sequence and $\alpha_1 \neq 0$, the elements x, y, a', b' form a regular sequence as well. But now by Lemma 2.11 the ideal $(x,y)^{e(R/Q)} + (a'x + b'y)$ is unmixed of multiplicity $e(R/Q)$ and by Lemma 2.10 it is equal to Q. (Note that since Q is primary to (x,y), $e(R/Q) = \lambda\big((R/Q)_{(x,y)}\big)$ and therefore $(x,y)^{e(R/Q)} \subset Q$.) Lemma 2.11 also asserts that $\mathrm{pd}(R/Q) \leqslant 3$.

If on the other hand $\alpha_i = 0$ for all $i = 1 \dots k$, then $(p_1, \dots, p_k) \subset (x,y)^2$ and by (3.8) the cubics p_i can be expressed entirely in terms of $3k+2$ linear forms l_{ij}, x, y. Note that the same holds when $\deg(ax+by) \geqslant 4$. We also find ourselves in a similar situation when $\deg(ax+by) = 2$. Namely, the cubics p_i are then contained in an ideal generated by four quadrics $x^2, xy, y^2, ax+by$ and so they can be expressed entirely in terms of $4k+4$ linear forms $l_{i1}, l_{i2}, l_{i3}, l_{i4}, x, y, a, b$ with $i = 1 \dots k$. □

We now establish a bound of 20 for the projective dimension of $R/(f,g,h)$ in the case of multiplicity 5. Let p_1, p_2 be any two cubic forms in the unmixed part I of (f,g,h) which form a regular sequence and let I' denote the link $(p_1,p_2) : I$. We have $e(R/I') = 9 - 5 = 4$. By the associativity formula (2.3) there are eleven possible types for the link I', namely:

$$\begin{aligned} &\langle e=4; \lambda=1\rangle, && \langle e=1; \lambda=4\rangle, \\ &\langle e=1,3; \lambda=1,1\rangle, && \langle e=1,1; \lambda=1,3\rangle, \\ &\langle e=2,2; \lambda=1,1\rangle, && \langle e=1,1; \lambda=2,2\rangle, \\ &\langle e=1,1,2; \lambda=1,1,1\rangle, && \langle e=1,1,1; \lambda=1,1,2\rangle, \\ &\langle e=2; \lambda=2\rangle, && \langle e=1,2; \lambda=2,1\rangle, \\ &\langle e=1,1,1,1; \lambda=1,1,1,1\rangle. && \end{aligned}$$

The general argument which we are about to enter consists of the following parts:

- Either the link I' contains a quadric form, in which case $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12,
- or we give a bound for the projective dimension of R/I', which in turn bounds (by one more) the projective dimension of $R/(f,g,h)$ by Theorem 3.3,
- or, by drawing on Remark 3.13 or by exhibiting that I' is contained in an ideal generated by a set of given quadrics, we show that the cubics p_1 and p_2 can be expressed entirely in terms of (at most) 12 linear forms, whereas any one cubic in I' requires (at most) 8 linear forms.

Recall that p_1 and p_2 are two arbitrary cubics in I' which form a regular sequence. So, unless we are able to obtain a bound for $\mathrm{pd}(R/(f,g,h))$ from the first two parts of the above argument, we apply the third part to the choice of, say, f, g and then to h and thus place the cubics f, g, h inside a subring generated by no more than 12+8 linear forms. Hence $\mathrm{pd}(R/(f,g,h)) \leqslant 20$.

$\underline{\langle e=4; \lambda=1\rangle}$ If I' contains a quadric form, then $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. So suppose I' does not contain any quadrics; in particular, I' is non-degenerate and it is not a complete intersection. By part (c) of Theorem 2.18, I' is the defining ideal of a generic projection of the Veronese surface $V_5 \subset \mathbb{P}^5$ and $\mathrm{pd}(R/I') = 4$. Thus, $\mathrm{pd}(R/(f,g,h)) \leqslant 5$ by Theorem 3.3.

We point out that the bound of 5 obtained in this case is in fact sharp. We will demonstrate this by constructing an example in Section 3.3.

$\underline{\langle e=1; \lambda=4\rangle}$ I' is primary to (x,y) with independent linear forms x, y such that $\lambda\big((R/I')_{(x,y)}\big) = 4$. So the Hilbert function of $(R/I')_{(x,y)}$ is either $(1,2,1)$ or $(1,1,1,1)$.

First suppose $(R/I')_{(x,y)}$ has Hilbert function $(1,2,1)$. Then the Hilbert function of $\big({}^{R}/_{I':(x,y)}\big)_{(x,y)}$ is given by either $(1,1)$ or $(1,2)$, depending on whether or not $(R/I')_{(x,y)}$ has a socle element outside $(x,y)^2_{(x,y)}$.

If $\left({}^{R}/_{I':(x,y)}\right)_{(x,y)}$ has Hilbert function $(1,2)$, then $I' : (x,y) = (x,y)^2$ and since $I' \subset I' : (x,y) = (x^2, xy, y^2)$, the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 8 linear forms.

If on the other hand $\left({}^{R}/_{I':(x,y)}\right)_{(x,y)}$ has Hilbert function $(1,1)$, then, by Proposition 2.13, $I' : (x,y) = (x,y)^2 + (ax+by)$ with elements a, b such that x, y, a, b form a regular sequence. If the term $ax + by$ is linear, then I' contains quadrics — such as $(ax+by)x$ and $(ax+by)y$ — and $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. And if $\deg(ax+by) \geqslant 2$, then we are done by Remark 3.13. That is, either $\mathrm{pd}(R/I') \leqslant 3$ and $\mathrm{pd}(R/(f,g,h)) \leqslant 4$, or the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

Now suppose $(R/I')_{(x,y)}$ has Hilbert function $(1,1,1,1)$. Then $\left({}^{R}/_{I':(x,y)^2}\right)_{(x,y)}$ has Hilbert function $(1,1)$ and so, by Proposition 2.13, $I' : (x,y)^2 = (x,y)^2 + (ax+by)$ with elements a, b such that x, y, a, b form a regular sequence.

If $\deg(ax+by) = 1$, then we relabel the term $ax+by$ as x so that $I' : (x,y)^2 = (x, y^2)$. In particular, $x(x,y)^2 = (x^3, x^2y, xy^2) \subset I'$. Since $(R/I')_{(x,y)}$ has Hilbert function $(1,1,1,1)$, I' must also contain a generator of the form $cx + dy$ with $(c,d) \not\subset (x,y)$.

Multiplying $cx + dy$ with y^2 and reducing it modulo xy^2, we see that $dy^3 \in I'$. As $(R/I')_{(x,y)}$ has Hilbert function $(1,1,1,1)$, we cannot have $(x,y)^3 \subseteq I'$. But I' already contains (x^3, x^2y, xy^2). So $y^3 \notin I'$ and therefore $d \in (x,y)$. (Recall that I' is primary to (x,y).) In particular, $dxy \in (x^2y, xy^2) \subset I'$. Multiplying $cx + dy$ with x and reducing it modulo dxy, we see that $cx^2 \in I'$. As $(c,d) \not\subset (x,y)$ and $d \in (x,y)$, we have $c \notin (x,y)$ and so $x^2 \in I'$. Thus, I' contains a quadric and $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12.

And if $\deg(ax+by) \geqslant 2$, then we apply Remark 3.13 again: Either $\mathrm{pd}(R/I') \leqslant 3$ and $\mathrm{pd}(R/(f,g,h)) \leqslant 4$, or the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

$\underline{\langle e = 1,3; \lambda = 1,1\rangle}$ $I' = (x,y) \cap P$ with independent linear forms x, y and a height two prime ideal P of multiplicity 3. If P contains a linear form l, then I' contains a quadric — such as xl or yl — and $\mathrm{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12.

If on the other hand P is non-degenerate, then, as outlined in Section 2.4.1, P is the ideal of 2×2 minors of a 3×2 matrix of indeterminates, that is, P is generated by three quadrics in at most six variables. As $I' \subset P$, the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

$\underline{\langle e = 1,1; \lambda = 1,3 \rangle}$ $I' = (u, v) \cap I_3$ with independent linear forms u, v and an ideal I_3 of type $\langle e = 1; \lambda = 3 \rangle$. That is, I_3 is primary to (x, y) with independent linear forms x, y and $\lambda\big((R/I_3)_{(x,y)}\big) = 3$. In particular, $(x, y)^3 \subset I_3$ and the Hilbert function of $(R/I_3)_{(x,y)}$ is either $(1, 2)$ or $(1, 1, 1)$. We may assume that $\operatorname{ht}(x, y, u, v) = 3$, for otherwise $I' \subset (u, v) \cap (x, y) = (xu, xv, yu, yv)$ and the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms. So, without loss of generality, we may replace u by x and write $I' = (x, v) \cap I_3$.

If $(R/I_3)_{(x,y)}$ has Hilbert function $(1, 2)$, then $I_3 = (x, y)^2$ and so $I' = (x^2, xy, y^2v)$. It is easily seen that in this case R/I' is Cohen-Macaulay (and consequently, so is R/I). Thus, $\operatorname{pd}(R/I') = 2$ and $\operatorname{pd}(R/(f, g, h)) \leqslant 3$ by Theorem 3.3.

If on the other hand $(R/I_3)_{(x,y)}$ has Hilbert function $(1, 1, 1)$, then $I_3 : (x, y)$ is of type $\langle e = 1; \lambda = 2 \rangle$. By Proposition 2.13 we have $I_3 : (x, y) = (x, y)^2 + (ax + by)$ with elements a, b such that $\operatorname{ht}(x, y, a, b) > 3$.

If $\deg(ax+by) = 1$, then we may relabel $ax+by$ as x so that $I_3 : (x, y) = (x, y^2)$. It now follows from Lemma 2.15 that I_3 is of the form $(x^2, xy, y^3, cx+dy^2)$. In particular, modulo (x, v) the ideal I_3 is two-generated: $(x, v) + I_3 = (x, v) + (y^3, dy^2)$. To bound the projective dimension of R/I', we consider the short exact sequence

$$0 \longrightarrow \frac{R}{I'} \longrightarrow \underbrace{\frac{R}{(x,v)} \oplus \frac{R}{I_3}}_{\text{proj. dim.} \leqslant 3} \longrightarrow \underbrace{\frac{R}{(x,v,y^3,dy^2)}}_{\text{proj. dim.} \leqslant 4} \longrightarrow 0 \tag{3.10}$$

and note that by Lemma 2.12 the middle term has projective dimension at most 3. To bound the projective dimension of the right term $R/(x, v, y^3, dy^2)$, we exploit the fact that x, v are linear forms and write the ideal (x, v, y^3, dy^2) as $(x, v, \tilde{y}^3, \tilde{d}\tilde{y}^2)$, where $(\ \ )^{\sim}$ denotes the reduction of an element modulo x and v. This has the effect that now x, v

form a regular sequence modulo $(\tilde{y}^3, \tilde{d}\tilde{y}^2)$ and so

$$\begin{aligned}\operatorname{pd}\left({}^{R}\!/\!_{(x,v,y^3,dy^2)}\right) &= \operatorname{pd}\left({}^{R}\!/\!_{(x,v,\tilde{y}^3,\tilde{d}\tilde{y}^2)}\right) \\ &= \operatorname{pd}\left({}^{R}\!/\!_{(\tilde{y}^3,\tilde{d}\tilde{y}^2)}\right) + 2 \ \leqslant\ 4.\end{aligned}$$

Applying Lemma 2.1 to the short exact sequence (3.10) now yields $\operatorname{pd}(R/I') \leqslant 3$ and so $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.3.

If $\deg(ax+by) \geqslant 2$, then we apply the argument of Remark 3.13 to the ideal I_3. That is, unless the cubics $p_1, p_2 \in I' \subset I_3$ can be expressed entirely in terms of 12 linear forms, we have $I_3 = (x,y)^3 + (a'x + b'y)$. As above, we observe that modulo (x,v) the ideal I_3 is two-generated: $(x,v) + I_3 = (x,v) + (y^3, b'y)$. So we have a short exact sequence similar to (3.10)

$$0 \longrightarrow \frac{R}{I'} \longrightarrow \frac{R}{(x,v)} \oplus \frac{R}{I_3} \longrightarrow \frac{R}{(x,v,y^3,b'y)} \longrightarrow 0$$

in which the middle term has projective dimension at most 3 by Lemma 2.11. Thus, the same arguments as in the case $\deg(ax+by) = 1$ above, verbatim, yield $\operatorname{pd}(R/I') \leqslant 3$ and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$.

$\underline{\langle e = 2,2; \lambda = 1,1\rangle}$ $I' = (l_1, q_1) \cap (l_2, q_2)$ with linear forms l_1, l_2 and irreducible quadrics q_1, q_2. As I' contains the quadric $l_1 l_2$, we have $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12.

$\underline{\langle e = 1,1; \lambda = 2,2\rangle}$ By Proposition 2.13, $I' = (x^2, xy, y^2, ax+by) \cap (u^2, uv, v^2, cu+dv)$ where x,y,u,v are linear forms and $\operatorname{ht}(x,y,u,v) = 3$ or 4. If $\operatorname{ht}(x,y,u,v) = 3$, then, without loss of generality, we may replace u by x. In this case I' contains the quadric x^2 and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. If on the other hand $\operatorname{ht}(x,y,u,v) = 4$, then $I' \subset (x,y) \cap (u,v) = (xu, xv, yu, yv)$. So the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

$\underline{\langle e = 1,1,2; \lambda = 1,1,1\rangle}$ $I' = (x,y) \cap (u,v) \cap (l,q)$ with linear forms x,y,u,v,l and an irreducible quadric q. If $\operatorname{ht}(x,y,u,v) = 3$, then, without loss of generality, we may replace u by x and write $I' = (x, yv) \cap (l,q)$. In this case I' contains the quadric xl and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. If on the other hand $\operatorname{ht}(x,y,u,v) = 4$, then $I' \subset (x,y) \cap (u,v) = (xu, xv, yu, yv)$ and the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

$\underline{\langle e = 1,1,1; \lambda = 1,1,2\rangle}$ By Proposition 2.13, I' admits a primary decomposition of the form $I' = (u,v) \cap (s,t) \cap (x^2, xy, y^2, ax+by)$ with linear forms u,v,s,t,x,y. If $\operatorname{ht}(u,v,s,t) = 4$, then $I' \subset (u,v) \cap (s,t) = (us, ut, vs, vt)$ and the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 12 linear forms.

If on the other hand $\operatorname{ht}(u,v,s,t) = 3$, then, without loss of generality, $u = s$ and $I' = (u, vt) \cap (x^2, xy, y^2, ax+by)$. Note that if $u \in (x,y)$, then I' contains the quadric u^2 and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. So we may further assume that $\operatorname{ht}(u,x,y) = 3$. We now use the inclusion $I' \subset (u,vt) \cap (x,y)$ to bound the number of linear forms needed to write p_1 and p_2.

If $vt \notin (x,y)$, then $I' \subset (ux, uy, vtx, vty)$ and the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of 9 linear forms. If on the other hand $vt \in (x,y)$, then either $v \in (x,y)$ or $t \in (x,y)$, for (x,y) is a prime ideal. Say $v \in (x,y)$ and, without loss of generality, relabel v as x. Now $I' \subset (ux, uy, xt)$ and $p_1, p_2 \in I'$ can be expressed entirely in terms of 10 linear forms.

$\underline{\langle e = 2; \lambda = 2\rangle}$ I' is primary to a prime ideal $P = (l,q)$ with a linear form l and an irreducible quadric q such that $\lambda\big(R_P/I'_P\big) = 2$. Thus, locally at P, we must have $P_P^2 \subset I'_P$. But primary ideals are contracted ideals in the sense that $I' = I'R_P \cap R$. Hence $P^2 \subset I'$ globally. So I' contains the quadric l^2 and therefore $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12.

$\underline{\langle e = 1,2; \lambda = 2,1\rangle}$ By Proposition 2.13, I' admits a primary decomposition of the form $I' = (x^2, xy, y^2, ax+by) \cap (l,q)$ with linear forms x,y,l, an irreducible quadric q, and elements a,b such that $\operatorname{ht}(x,y,a,b) > 3$. If $l \in (x,y)$ or if $\deg(ax+by) = 1$, then I' contains the quadric l^2 or $(ax+by)l$, respectively, and $\operatorname{pd}(R/(f,g,h)) \leqslant 4$ by Theorem 3.12. So we may assume that $\operatorname{ht}(x,y,l) = 3$ and $\deg(ax+by) \geqslant 2$, that is, x,y,l and x,y,a,b are both regular sequences.

As laid out in the proof of Remark 3.13, we may further reduce to the case where $\deg(ax+by) = 3$ and $ax+by = p_1$. (Recall that I' is linked to $I = (f,g,h)^{\text{unm}}$ by two cubic forms p_1 and p_2, that is, $I' = (p_1,p_2) : I$.) Indeed, if $\deg(ax+by) = 2$ or $\geqslant 4$, then the cubics $p_1, p_2 \in I'$ can be expressed entirely in terms of (at most) 12 linear forms. The same holds when $\deg(ax+by) = 3$ as long as $(p_1, p_2) \subset (x, y)^2$. And if $\deg(ax+by) = 3$ and one of the cubics, say p_1, has a non-zero contribution from the term $ax+by$, then we may replace $ax+by$ by p_1 without changing the ideal $(x,y)^2 + (ax+by)$ — see (3.9) et seq. on page 68. So without loss of generality $ax + by = p_1$.

Having replaced the cubic $ax+by$ by p_1, we may no longer assume that a and b are reduced modulo (x, y). However, as $p_1 \in I'$, we now have $ax+by \in (l, q)$, say $ax + by = cl + l'q$ with a quadric c and a linear form l'. This reduces the challenge of having to deal with three quadrics a, b, and q to that of having to deal with only two quadrics c and q. By Lemma 3.8 we have

$$I' \;=\; \big[(x,y)^2 \cap (l,q)\big] + (cl + l'q) \;\subset\; (x,y) \cap (l,q).$$

To bound the projective dimension of R/I', first suppose $q \in (x, y)$, say $q = l_1x+l_2y$ with linear forms l_1, l_2. Since $cl + l'q \in (x, y)$, it follows that $cl \in (x, y)$ and as x, y, l form a regular sequence, we must have $c \in (x, y)$, say $c = l_3x + l_4y$ with linear forms l_3, l_4. Now we can place the generators of I' inside the subring $k[x, y, l, l', l_1, l_2, l_3, l_4]$. So $\mathrm{pd}(R/I') \leqslant 8$ and $\mathrm{pd}(R/(f, g, h)) \leqslant 9$ by Theorem 3.3.

Now suppose $q \notin (x, y)$. Since we may reduce q modulo l without changing the ideal (l, q), this is tantamount to $q \notin (x, y, l)$, that is, x, y, l, q form a regular sequence. Thus, from $ax + by = cl + l'q$ we glean $c \in (x, y, q)$, say $c = l_1x + l_2y + \alpha q$ with linear forms l_1, l_2 and a scalar $\alpha \in k$. This places the generators of I' inside the subring $k[x, y, l, l', l_1, l_2, q]$. Let L denote the ideal generated by the linear forms x, y, l, l', l_1, l_2.

If $q \notin L$, then the generators of L along with q form a regular sequence of length at most 7, in which case $\mathrm{pd}(R/I') \leqslant 7$ and $\mathrm{pd}(R/(f, g, h)) \leqslant 8$. If on the other hand $q \in L$, then q can be expressed in terms of the generators of L using no more than six additional linear forms, in which case $\mathrm{pd}(R/I') \leqslant 12$ and $\mathrm{pd}(R/(f, g, h)) \leqslant 13$.

$\underline{\langle e = 1, 1, 1, 1; \lambda = 1, 1, 1, 1\rangle}$ I' is the intersection of four height two prime ideals, each of which is generated by two linear forms. Clearly, $\mathrm{pd}(R/I') \leqslant 8$ and therefore $\mathrm{pd}(R/(f, g, h)) \leqslant 9$ by Theorem 3.3.

3.2.7 Multiplicity Six

Using linkage and Theorem 3.3 as our main tools, we give a bound of 20 for the projective dimension of $R/(f,g,h)$ in the case of multiplicity 6. Let p_1, p_2 be any two cubic forms in the unmixed part I of (f,g,h) which form a regular sequence and let I' denote the link $(p_1,p_2) : I$. We have $e(R/I') = 9 - 6 = 3$. By the associativity formula (2.3) there are five possible types for the link I', namely:

$$\langle e = 3; \lambda = 1 \rangle, \qquad \langle e = 1; \lambda = 3 \rangle,$$
$$\langle e = 1,2; \lambda = 1,1 \rangle, \qquad \langle e = 1,1; \lambda = 1,2 \rangle,$$
$$\langle e = 1,1,1; \lambda = 1,1,1 \rangle.$$

In what follows we consider each of the above cases and either exhibit a bound for the projective dimension of R/I', and thereupon for that of $R/(f,g,h)$, or we infer that the cubics f,g,h are contained in an ideal generated by a known number of quadrics which are expressed in terms of a fixed number of linear forms.

$\underline{\langle e = 3; \lambda = 1 \rangle}$ As shown in Section 2.4.1, R/I' is Cohen-Macaulay (and consequently, so is R/I). Thus, $\mathrm{pd}(R/I') = 2$ and $\mathrm{pd}(R/(f,g,h)) \leqslant 3$ by Theorem 3.3.

$\underline{\langle e = 1,2; \lambda = 1,1 \rangle}$ $I' = (x,y) \cap (l,q)$ with linear forms x,y,l and an irreducible quadric q. We have shown previously that either $\mathrm{ht}(x,y,l,q) = 3$ and R/I' is Cohen-Macaulay, or $\mathrm{ht}(x,y,l,q) = 4$ and $\mathrm{pd}(R/I') = 3$ — see case $\langle e = 1,2; \lambda = 1,1 \rangle$ on page 51 in Section 3.2.3. Hence $\mathrm{pd}(R/(f,g,h)) \leqslant 4$.

$\underline{\langle e = 1,1,1; \lambda = 1,1,1 \rangle}$ I' is the intersection of three height two prime ideals, each of which is generated by two linear forms. Clearly, $\mathrm{pd}(R/I') \leqslant 6$ and therefore $\mathrm{pd}(R/(f,g,h)) \leqslant 7$ by Theorem 3.3.

$\underline{\langle e = 1,1; \lambda = 1,2 \rangle}$ By Proposition 2.13, I' admits a primary decomposition of the form $(u,v) \cap (x^2, xy, y^2, ax+by)$ with independent linear forms u,v, independent linear forms x,y, and elements a,b such that $\mathrm{ht}(x,y,a,b) > 3$. As so often, we study this intersection via the short exact sequence

$$0 \to \frac{R}{I'} \to \underbrace{\frac{R}{(u,v)} \oplus \frac{R}{(x,y)^2+(ax+by)}}_{\text{projective dimension } \leqslant 3} \to \frac{R}{(u,v)+(x,y)^2+(ax+by)} \to 0 \qquad (3.11)$$

in which the projective dimension of the middle term is at most 3 by Lemma 2.11. To bound the projective dimension of the right term, we rewrite $(x,y)^2+(ax+by)$ as $(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})$, where $(\ )\tilde{}$ denotes the reduction of an element modulo the linear forms u and v. In other words, we remove any multiples of u and v from x, y, a, b. While this does not change the ideal $(u,v)+(x,y)^2+(ax+by)$, it has the effect that now u, v form a regular sequence modulo $(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})$ and so

$$\begin{aligned} \operatorname{pd}\big({}^{R}/_{(u,v)+(x,y)^2+(ax+by)}\big) &= \operatorname{pd}\Big({}^{R}/_{(u,v)+(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})}\Big) \\ &= \operatorname{pd}\Big({}^{R}/_{(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})}\Big)+2. \qquad (3.12) \end{aligned}$$

Next we bound the projective dimension of $R/\big((\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})\big)$.

Clearly, $\tilde{x}$ and $\tilde{y}$ are still linear forms; however, they may no longer be linearly independent. (In fact, one of them may be zero.) If $\tilde{x}$ and $\tilde{y}$ are linearly dependent, then the ideal $(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})$ is generated by two elements and therefore the projective dimension of $R/\big((\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})\big)$ is at most 2. If on the other hand $\tilde{x},\tilde{y}$ are linearly independent, then either $(\tilde{a},\tilde{b}) \subseteq (\tilde{x},\tilde{y})$ or $(\tilde{a},\tilde{b}) \not\subseteq (\tilde{x},\tilde{y})$. In the former case $(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y}) = (\tilde{x},\tilde{y})^2$ and $\operatorname{pd}\big({}^{R}/_{(\tilde{x},\tilde{y})^2}\big) \leqslant 2$. And in the latter case we have $\operatorname{pd}\Big({}^{R}/_{(\tilde{x},\tilde{y})^2+(\tilde{a}\tilde{x}+\tilde{b}\tilde{y})}\Big) \leqslant 3$ by Lemma 2.11.

Thus, by (3.12), $\operatorname{pd}\big({}^{R}/_{(u,v)+(x,y)^2+(ax+by)}\big) \leqslant 5$. Finally, applying Lemma 2.1 to the short exact sequence (3.11) yields $\operatorname{pd}(R/I') \leqslant 4$ and therefore $\operatorname{pd}(R/(f,g,h)) \leqslant 5$ by Theorem 3.3.

$\underline{\langle e=1; \lambda=3\rangle}$ I' is primary to (x,y), where x,y are independent linear forms, and $\lambda\big((R/I')_{(x,y)}\big) = 3$. Either $I' = (x,y)^2$ or, locally at (x,y), the Hilbert function of $(R/I')_{(x,y)}$ is given by $(1,1,1)$. In the former case R/I' (and R/I) are Cohen-Macaulay and $\operatorname{pd}(R/(f,g,h)) \leqslant 3$ by Theorem 3.3. In the latter case Proposition 2.13 yields that $I' : (x,y) = (x,y)^2+(ax+by)$ with elements a,b such that $\operatorname{ht}(x,y,a,b) > 3$.

Recall that $I' = (p_1, p_2) : I$. Thus, we have an inclusion

$$(p_1, p_2) \subset I' \subset I' : (x, y) = (x, y)^2 + (ax + by)$$

for any two cubics p_1, p_2 in the unmixed part I of (f, g, h) which form a regular sequence. (Here the elements x, y, a, b depend on the particular choice of the cubics p_1 and p_2.) We now give a bound for $\mathrm{pd}(R/(f, g, h))$ by considering the degree of the term $ax + by$.

If $\deg(ax + by) = 1$ for some choice of p_1 and p_2, then we may relabel the term $ax + by$ as x and write $I' : (x, y) = (x, y^2)$. By Lemma 2.15, $I' = (x^2, xy, y^3, cx + dy^2)$ with elements c and d such that $\mathrm{ht}(x, y, c, d) > 3$. By Lemma 2.12, $\mathrm{pd}(R/I') \leqslant 3$ and by Theorem 3.3, $\mathrm{pd}(R/(f, g, h)) \leqslant 4$.

If $\deg(ax + by) \geqslant 2$ for some choice of p_1 and p_2, then we are in the position to invoke an argument which was already used in Section 3.2.6. By Remark 3.13, either $\mathrm{pd}(R/I') \leqslant 3$ and consequently $\mathrm{pd}(R/(f, g, h)) \leqslant 4$, or the cubics p_1, p_2 can be expressed in terms of 12 linear forms.

So, unless $\mathrm{pd}(R/(f, g, h)) \leqslant 4$, we know that *every* two cubics $p_1, p_2 \in I$ which form a regular sequence can be expressed entirely in terms of 12 linear forms, while any arbitrary cubic in I can be expressed entirely in terms of 8 linear forms. Thus, f, g, h can be written entirely in terms of 20 linear forms and $\mathrm{pd}(R/(f, g, h)) \leqslant 20$.

3.3 Three Cubics of Projective Dimension 5

In this section we construct an ideal generated by three cubic forms f, g, h of height two and multiplicity 5 such that $\mathrm{pd}(R/(f, g, h)) = 5$.

Our starting point is $I(V_5)$, the defining ideal of the Veronese surface $V_5 \subset \mathbb{P}^5$. (See Sections 2.4.2 and 2.4.3.) Note that $\mathrm{ht}(I(V_5)) = 3$. In order to obtain an ideal of height two, we project V_5 from a general point of $\mathbb{P}^5$ onto $\mathbb{P}^4$ and, consistent with our notation in the preceding section, we denote the defining ideal of the resulting variety by I'. By part (c) of Theorem 2.18, I' is generated by seven cubic forms and $\mathrm{pd}(R/I') = 4$. We propose that I' is linked to the unmixed part I of an ideal generated by three cubic forms f, g, h. If so, then it follows from Proposition 3.4 that $\mathrm{pd}(R/(f,g,h)) = \mathrm{pd}(R/I') + 1 = 5$.

To construct an ideal I which is linked to I', we choose two generic cubics p_1, p_2 in I' and set $I := (p_1, p_2) : I'$. In the calculation carried out below using the software Macaulay 2 [M2], the resulting ideal I is generated by five cubic forms. Choosing three cubic forms f, g, h among the generators of I (or three generic linear combinations of them) yields an ideal with $(f,g,h)^{\mathrm{unm}} = I$ and hence $\mathrm{pd}(R/(f,g,h)) = 5$.

$$\begin{array}{ccccc} & e=4 & & e=5 & \\ I(V_5) \rightsquigarrow & I' & & I & \supset (f,g,h) \\ & & \diagdown \qquad \diagup & & \\ & & (p_1,p_2) & & \\ & & e=9 & & \end{array}$$

```
Macaulay 2, version 0.9.2-20
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : R = QQ[x_0..x_5];

i2 : veronese = trim minors(2, genericSymmetricMatrix(R, x_0, 3))

             2                                    2
o2 = ideal (x  - x x , x x  - x x , x x  - x x , x  - x x , x x  - x x , ... )
             4    3 5   2 4    1 5   2 3    1 4   2    0 5   1 2    0 4

o2 : Ideal of R

i3 : Rbar = R/veronese;

i4 : S = QQ[y_0..y_4];

i5 : link = kernel map(Rbar, S, random(Rbar^{1}, Rbar^5))

             2     1188901803024280156
o5 = ideal (y y  - -------------------*y y y  - ... )
             0 3    929813254718426775  0 1 3

o5 : Ideal of S
```

```
i6 : degrees link

o6 = {{3}, {3}, {3}, {3}, {3}, {3}, {3}}

o6 : List

i7 : p1p2 = ideal(mingens link * random(S^7, S^2));

o7 : Ideal of S

i8 : time   unmix = p1p2 : link
     -- used 17.13 seconds

             2     30913749037933699580897677049984495603 49
o8 = ideal (y y  + ---------------------------------------*y y y  + ... )
             0 2    21259060860591108655737223200544528540 7  0 1 2

o8 : Ideal of S

i9 : degrees unmix

o9 = {{3}, {3}, {3}, {3}, {3}}

o9 : List

i10 : fgh = ideal(unmix_0, unmix_1, unmix_2)

              2     3091374903793369958089767704998449560349
o10 = ideal (y y  + ----------------------------------------*y y y  + ... )
              0 2    2125906086059110865573722320054452854 07  0 1 2

o10 : Ideal of S

i11 : betti res fgh

o11 = total: 1 3 8 10 5 1
          0: 1 . .  . . .
          1: . . .  . . .
          2: . 3 .  . . .
          3: . . .  . . .
          4: . . 8 10 5 1
```

3.4 Further Directions

As a continuation of this work, it seems worthwhile to attempt to improve upon the bound of 36 for three cubic forms, as this bound is most likely not optimal. In fact, there are no known examples of three cubics with projective dimension greater than 5.

Another question arose during an attempt to weaken the degeneracy assumption of Corollary 3.2 and thereby expose a larger class of three-generated ideals with projective dimension $\leqslant 3$. As it stands, Corollary 3.2 can be stated in the following, seemingly more general form:

If the unmixed part I of a height two, three-generated ideal $J \subset R$ contains an element f such that $R/(f)$ is a unique factorization domain (UFD), then $\mathrm{pd}(R/J) \leqslant 3$.

Note that no assumptions are made on the degrees of the generators of J. The above formulation leads to the following question: Under what circumstances can one infer the existence of an element $f \in I$ such that $R/(f)$ is a again UFD? For instance, if f is a quadric, then it is known that $R/(f)$ is a UFD if f has rank $\geqslant 5$.

The results in Section 2.3 lead to the following question: Is there a bound on the projective dimension of unmixed ideals of given height and multiplicity, such as, say, unmixed ideals of height two and multiplicity three? Is it possible to give a structure theorem for such ideals, as in Proposition 2.13?

Lastly, can Question 1.1 be reduced to three-generated ideals? This thought is motivated by the following

Theorem 3.14 (Bruns [B1])**.** *Let M be a finitely generated R-module and let*

$$\mathsf{F}: \quad 0 \to F_n \xrightarrow{f_n} \cdots \xrightarrow{f_4} F_3 \xrightarrow{f_3} F_2 \to F_1 \to F_0$$

be a free resolution of M. Set $r := \mathrm{rank}(f_3)$. Then there exist homomorphisms f_3', f_2', f_1' with $\mathrm{Im}(f_3') \cong \mathrm{Im}(f_3)$ such that the sequence

$$\mathsf{F}': \quad 0 \to F_n \xrightarrow{f_n} \cdots \xrightarrow{f_4} F_3 \xrightarrow{f_3'} R^{r+2} \xrightarrow{f_2'} R^3 \xrightarrow{f_1'} R$$

is exact. That is, F' is a free resolution of a three-generated ideal.

In other words, can Theorem 3.14 be used to extrapolate the results for three-generated ideals to the general case with an arbitrary number of generators? Early considerations indicate that for this purpose, one might have to not only appeal to the data on the generators of the ideal J, but also to those of higher syzygies of R/J. If so, the question becomes: For what $k \geqslant 1$ do the modules $F_i \cong \bigoplus_{j=1}^{N_i} R(-d_{ij})$, $i = 1 \dots k$, determine a bound on the length of any minimal graded free resolution of the form $\cdots \to F_2 \to F_1 \to R$?

In this context the following question was raised, which is of similar nature: It is known that, in general, bounds on the Castelnuovo-Mumford regularity of an ideal are doubly exponential in terms of the maximal degree of a minimal generator and the number of variables. Is it possible to give a polynomial bound for the regularity of the ideal if in addition, for some $k \geqslant 2$, the degrees of the minimal generators of the first k syzygies are known as well?

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