My research interests lie mainly in partial differential equations (PDE) and numerical analysis. More precisely, I have been working on the following two subjects: first the study of the asymptotic behavior of some singular perturbation problems in a smooth (curved) bounded domain including the Navier-Stokes equations, and second, some central issues of the finite volume discretization method, namely, the convergence and consistency of the cell-centered finite volume method when we discretize a two-dimensional bounded domain.

1. Accomplishments during Ph.D. study

1.1. Boundary layer analysis in a curved domain. From both the theoretical and computational points of view, boundary layers of the singularly perturbed problems have been well-studied; see, e.g., [Lio73, O’Ma77, VL57] for the general theory and [CT02, JT05] for computational applications. In particular, when the boundary is either characteristic or non-characteristic, asymptotic behavior of the Navier-Stokes (or Stokes) equations with small viscosity is studied in, e.g., [TW95, TW96, TW97-1, TW97-2, TW02, HT07, HT08]. However much remains to be done, and the study of small viscosity flows remains a major open problem.

During my graduate study, under the supervision of Professor R. Temam, we generalize the above mentioned works by considering a smooth (curved) domain instead of a domain with a flat boundary. To understand and resolve the geometrical difficulty from a curved boundary, we first consider, in [1, 3], the reaction-diffusion and heat equations in a curved domain when a small parameter \( \varepsilon > 0 \) is placed in front of the Laplacian. Due to the curvature on the boundary, the usual expansion in powers of \( \varepsilon \) does not give a suitable approximation. Indeed the usual expansion has to be adapted by introducing terms of order \( \varepsilon^{j+1/2}, j \geq 0 \), in the expansion. These terms appear because, for a curved domain, the normal direction changes along the boundary. Using the curvilinear coordinates adapted to the boundary for both the reaction-diffusion and heat problems, we find explicit expressions of the correctors at all orders and obtain the optimal convergence rates between the exact and approximate solutions.

In [4], we also study, in a curved domain with a characteristic boundary, the asymptotic behavior of solutions to the Stokes equations when the viscosity \( \varepsilon > 0 \) is small. Using the curvilinear system again, we show that the solutions of the Stokes equations behave like the corresponding linearized Euler solution except in a thin region near the boundary, where a certain heat solution is added as a corrector.

1.2. Finite volume method. Finite volumes (FV) are widely used both in engineering (see, e.g., [Chu02, HK98, SGN06]) and in geophysical fluid dynamics (GFD) (see, e.g., [MKL08, AP06, FLT08]), because of their local conservation property on each control
volume. In the mathematical analysis of the FV methods, one specific difficulty is due to the “weak consistency” of the FV methods; see, e.g., [DE06, EGH00, EGH06, FPT06, Sül91]. More precisely, the companion discrete FV derivative arising in the discrete integration by parts does not usually converge strongly to the corresponding derivative of the limit function. To overcome this difficulty, in [2], we follow the approach in [FPT06]. To compute the flux, however, we considered a different FV scheme, the so-called Taylor series expansion scheme (TSES). This scheme is more complicated than the scheme used in [FPT06], but more efficient at handling two-dimensional problems. The stability, convergence and consistency results of the FV method have been obtained for a rectangular domain \((0,1)^2\) in \(\mathbb{R}^2\) with rectangular meshes. In addition, as an application of the FV convergence result, we demonstrate how one can use the FV method to approximate weak solutions of some typical elliptic equations with Dirichlet boundary conditions, and show the convergence of such approximations via finite volumes to the weak solution of the original problem.

2. Recent work on the Boundary Layer analysis

As a post-doc at the University of California, Riverside, working in collaboration with my mentor, J. P. Kelliher, and others, I devoted myself to studying boundary layers of the Navier-Stokes equations at small viscosity. More precisely, in a 2 or 3D curved domain \(\Omega\) with smooth boundary \(\Gamma\), we consider the Navier-Stokes equations,

\[
\frac{\partial \mathbf{u}^\varepsilon}{\partial t} - \varepsilon \Delta \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} \quad \text{in } \Omega \times (0, T),
\]

\[
\text{div } \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega \times (0, T),
\]

\[
\mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,
\]

where \(\varepsilon > 0\) is the small viscosity parameter, \(T > 0\) is a fixed time, and the data \(\mathbf{f}\), and \(\mathbf{u}_0\) are assumed to be smooth.

Under various boundary conditions, the main task is to study the asymptotic behavior, as the viscosity \(\varepsilon\) tends to 0, of the Navier-Stokes solutions \(\mathbf{u}^\varepsilon\). The formal limit \(\mathbf{u}^0\) of \(\mathbf{u}^\varepsilon\) is a solution of the Euler equations,

\[
\frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla) \mathbf{u}^0 + \nabla p^0 = \mathbf{f} \quad \text{in } \Omega \times (0, T),
\]

\[
\text{div } \mathbf{u}^0 = 0 \quad \text{in } \Omega \times (0, T),
\]

\[
\mathbf{u}^0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T),
\]

\[
\mathbf{u}^0|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega.
\]

In the following subsections, we summarize some recent convergence results, appearing in [5, 6, 8, 9, P1], of \(\mathbf{u}^\varepsilon\) to \(\mathbf{u}^0\).

2.1. Navier-Stokes equations with generalized Navier boundary conditions. With J. P. Kelliher, in [5], we study the weak boundary layer phenomenon of the 3D Navier-Stokes equations (2.1), supplemented with the generalized Navier friction boundary conditions,

\[
\left\{ \begin{array}{l}
\mathbf{u}^\varepsilon \cdot \mathbf{n} = 0 \text{ on } \Gamma, \\
[S(\mathbf{u}^\varepsilon)\mathbf{n}]_{\text{tan}} + A \mathbf{u}^\varepsilon = 0 \text{ on } \Gamma,
\end{array} \right.
\]

where \(A\) is a matrix and \(S(\mathbf{u})\) is the stress tensor.
where \( S(u) = (\nabla u + (\nabla u)^\top)/2 \), \( n \) is the outer unit normal vector on \( \Gamma \), \( A \) is a type \((1, 1)\) tensor on the boundary having at least \( C^2 \)-regularity, and \( [\cdot]_{\text{tan}} \) denotes the tangential components of a vector on \( \Gamma \).

In coordinates on the boundary, \( A \) can be written in matrix form as \( A = (a_{ij})_{1 \leq i,j \leq 2} \), and hence, when \( A = \alpha I \), generalized Navier boundary conditions (2.3) reduce to the usual Navier friction boundary conditions,

\[
\begin{align*}
u^\varepsilon \cdot n &= 0, \quad [S(u^\varepsilon)n + \alpha u^\varepsilon]_{\text{tan}} = 0 \quad \text{on } \Gamma.
\end{align*}
\]

When \( A \) is the shape operator (Weingarten map) on \( \Gamma \), one obtains, as a special case, the boundary conditions,

\[
\begin{align*}
u^\varepsilon \cdot n &= 0, \quad (\text{curl } u^\varepsilon) \times n = 0 \quad \text{on } \Gamma,
\end{align*}
\]

which have been studied by several authors, e.g., [BC10, XX07].

Even with Navier boundary conditions there is a discrepancy between \( u^0 \) and \( u^\varepsilon \) on the boundary, so we expect boundary layer effects to occur. As first shown in [IP06], however, this boundary layer effect is mild enough to allow convergence of \( u^\varepsilon \) to \( u^0 \) in \( L^\infty(0,T;L^2(\Omega)) \) without using any artificial function correcting the difference \( u^\varepsilon - u^0 \) on the boundary. Later, the authors of [IS10] improve the convergence rate in [IP06] by using a corrector defined as a solution of a linearized Prandtl-type system of coupled equations.

We, on the other hand, use an asymptotic expansion of \( u^\varepsilon \) in the form \( u^\varepsilon \simeq u^0 + \theta^\varepsilon \), where the main part of the explicitly defined corrector \( \theta^\varepsilon \) exponentially decays from the boundary. Then, thanks to the regularity result of \( u^\varepsilon \) in [MR10], and using the anisotropic Agmon’s inequality, we prove the following convergence result of \( u^\varepsilon \) to \( u^0 \):

**Theorem 2.1.** Under some regularity assumptions on the data, as the viscosity \( \varepsilon \) tends to zero, \( u^\varepsilon \), a solution of the Navier-Stokes equations with generalized Navier boundary conditions converges to \( u^0 \), the solution of the Euler equations in the sense that

\[
\begin{align*}
\|u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(\Omega))} &\leq \kappa \varepsilon^{\frac{2}{3}}, \\
\|u^\varepsilon - u^0\|_{L^2(0,T;H^1(\Omega))} &\leq \kappa \varepsilon^{\frac{2}{3}}, \\
\|u^\varepsilon - u^0\|_{L^\infty([0,T] \times \Omega)} &\leq \kappa \varepsilon^{\frac{2}{3} - \delta},
\end{align*}
\]

for some \( 0 < \delta < 3/8 \), and a constant \( \kappa = \kappa(T, A, u_0, f) \), which are independent of \( \varepsilon \).

### 2.2. Navier-Stokes equations with non-characteristic boundaries

As a continuation and generalization of earlier work, [HT07] and [TW02], with M. Hamouda and R. Temam, in [6], we study the asymptotic behavior of the Navier-Stokes equations (2.1) when the smooth domain \( \Omega \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) is enclosed by \( \Gamma = \Gamma_O \cup \Gamma_I \), where \( \Gamma_O \) and \( \Gamma_I \) respectively denote “outer” and “inner” boundaries of \( \Omega \).

On each \( \Gamma_i \), \( i = O \) or \( I \), we assume that the velocity \( u^\varepsilon \) is known: \( u^\varepsilon|_{\Gamma_i} = U^i n \) where \( n \) is the unit normal vector to \( \Gamma \), pointing outward of \( \Omega \), with \( |U^i| > 0 \), \( i = O \) or \( I \). Then, using the conservation of mass equation, we supplement the Navier-Stokes equations (2.1) with the permeable boundary conditions,

\[
(2.5)
\begin{align*}
u^\varepsilon = \begin{cases} \\
U^O n & \text{on } \Gamma_O, \\
- U^I n & \text{on } \Gamma_I,
\end{cases}
\end{align*}
\]

where \( U^O|_{\Gamma_O} = U^I|_{\Gamma_I} = 0 \), \( U^O, U^I > 0 \).
The corresponding limit problem is the Euler system (2.2) with the boundary condition (2.2), replaced by

\[
\begin{cases}
\mathbf{u}_0 \cdot \mathbf{n} = U^0 \mathbf{n} & \text{on } \Gamma_O, \\
\mathbf{u}_0 = -U^I \mathbf{n} & \text{on } \Gamma_I.
\end{cases}
\]

(2.6)

One can verify that there exists a (smooth) vector field \( \mathbf{U} \) in \( \Omega \) satisfying

\[
\begin{align*}
\text{div } \mathbf{U} &= 0 \quad \text{in } \Omega, \\
\mathbf{U} &= U^0 \mathbf{n} \quad \text{on } \Gamma_O, \\
\mathbf{U} &= -U^I \mathbf{n} \quad \text{on } \Gamma_I.
\end{align*}
\]

Then setting

\[
v^\varepsilon := u^\varepsilon - U, \quad v^0 := u^0 - U,
\]

we rewrite the Navier-Stokes equations with (2.5), and the Euler equations with (2.6) in terms of \( v^\varepsilon \) and \( v^0 \):

\[
\begin{cases}
\frac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon + (U \cdot \nabla)v^\varepsilon + (v^\varepsilon \cdot \nabla)U + \nabla p^\varepsilon \\
= f + \varepsilon \Delta U - (U \cdot \nabla)U & \text{in } \Omega \times (0, T), \\
\text{div } v^\varepsilon &= 0 \quad \text{in } \Omega \times (0, T), \\
v^\varepsilon &= 0 \quad \text{on } \Gamma \times (0, T), \\
v^\varepsilon|_{t=0} &= v_0 := u_0 - U \quad \text{in } \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\frac{\partial v^0}{\partial t} + (v^0 \cdot \nabla)v^0 + (U \cdot \nabla)v^0 + (v^0 \cdot \nabla)U + \nabla p^0 \\
= f - (U \cdot \nabla)U & \text{in } \Omega \times (0, T), \\
\text{div } v^0 &= 0 \quad \text{in } \Omega \times (0, T), \\
v^0 \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_O \times (0, T), \\
v^0 &= 0 \quad \text{on } \Gamma_I \times (0, T), \\
v^0|_{t=0} &= v_0 \quad \text{in } \Omega.
\end{cases}
\]

Thanks to the curvilinear system adapted to the boundary, we construct a divergence-free corrector \( \Theta^0 \), which balances the discrepancy of \( v^\varepsilon \) and \( v^0 \) on the boundary \( \Gamma_O \), and obtain an asymptotic expansion of \( v^\varepsilon \) in the form,

\[
v^\varepsilon \simeq v^0 + \Theta^0, \quad \Theta^0 \simeq \theta^0 + \varepsilon \varphi^0.
\]

Any component of \( \theta^0 \) is an exponentially decaying function from the outer boundary \( \Gamma_O \), and \( \varphi^0 \) is a smooth vector field of unit order in any Sobolev space. Then, using the estimates on the corrector \( \theta^0 \), we obtain the following:

**Theorem 2.2.** Under regularity assumptions on the data, there exists \( T^0 > 0 \) such that

\[
\|v^\varepsilon - (v^0 + \theta^0)\|_{L^\infty(0,T^0;L^2(\Omega))} \leq \kappa \varepsilon, \quad \|v^\varepsilon - (v^0 + \theta^0)\|_{L^2(0,T^0;H^1(\Omega))} \leq \kappa \varepsilon^\frac{1}{2},
\]

for a constant \( \kappa = \kappa(\Omega, v_0, f, T^0) > 0 \), independent of \( \varepsilon \). Moreover, as the viscosity \( \varepsilon \) tends to zero, the Navier-Stokes solution \( v^\varepsilon \) converges to \( v^0 \) in the sense that

\[
\|v^\varepsilon - v^0\|_{L^\infty(0,T^0;L^2(\Omega))} \leq \kappa \varepsilon^\frac{1}{2}.
\]
In addition, in [6], we also obtain the asymptotic expansion of \( v^\varepsilon \) at the next order \( \varepsilon \), which improves the estimates in (2.7) by \( \varepsilon \).

2.3. Vorticity layers of the 2D Navier-Stokes equations with a slip type boundary condition. Generalizing an earlier work [JPT11], in [8], with C.-Y. Jung, we study the asymptotic behavior at small viscosity of the Navier-Stokes equations (2.1) where the 2D domain \( \Omega \) is smooth and simply connected. The Navier-Stokes equations are supplemented with the impermeable and slip boundary conditions,

\[
\begin{align*}
\mathbf{u}^\varepsilon \cdot \mathbf{n} &= 0, & \text{curl } \mathbf{u}^\varepsilon &= 0, & \text{on } \Gamma.
\end{align*}
\]

This set of boundary conditions is a 2D version of (2.4), in which the scalar friction coefficient \( \alpha \) is equal to two times the curvature on the boundary; see [Kel06]. Hence the convergence results in Theorem 2.1 are valid. In particular, as the viscosity goes to zero, we have the convergence of \( \mathbf{u}^\varepsilon \) to the Euler solution \( \mathbf{u}^0 \) in the sense that

\[
\begin{align*}
\| \mathbf{u}^\varepsilon - \mathbf{u}^0 \|_{L^\infty(0,T;L^2(\Omega))} &\leq C\varepsilon^{3/4}, \\
\| \mathbf{u}^\varepsilon - \mathbf{u}^0 \|_{L^2(0,T;H^1(\Omega))} &\leq C\varepsilon^{1/4},
\end{align*}
\]

for a constant \( C \), depending on the data, but independent of the viscosity \( \varepsilon \). (See related works, e.g., [LNP05, Mil07, WXZ12].)

The main goal of [8] is to improve the convergence result (2.9). Toward this end, we introduce the stream function \( \psi^\varepsilon \), and the scalar vorticity \( \omega^\varepsilon \) of \( \mathbf{u}^\varepsilon \) such that

\[
\begin{align*}
\nabla \perp \psi^\varepsilon &= \mathbf{u}^\varepsilon, & \omega^\varepsilon &= \text{curl } \mathbf{u}^\varepsilon,
\end{align*}
\]

where \( \nabla \perp = (\partial/\partial y, -\partial/\partial x) \). (In addition, we have \( \Delta \psi^\varepsilon = -\omega^\varepsilon \).) Then the 2D Navier-Stokes equations with (2.8) can be written in the vorticity from:

\[
\begin{align*}
\frac{\partial \omega}{\partial t} - \varepsilon \Delta \omega^\varepsilon + \frac{\partial \psi^\varepsilon}{\partial x} \frac{\partial \omega}{\partial x} - \frac{\partial \psi^\varepsilon}{\partial y} \frac{\partial \omega}{\partial y} &= f := \text{curl } \mathbf{f} & \text{in } \Omega \times (0,T), \\
\omega^\varepsilon &= \psi^\varepsilon = 0 & \text{on } \Gamma, \\
\omega^\varepsilon|_{t=0} &= \omega^0 := \text{curl } \mathbf{u}_0 & \text{in } \Omega.
\end{align*}
\]

Taking the scalar curl on the Euler equations (2.2), we find the corresponding limit problem,

\[
\begin{align*}
\frac{\partial \omega^0}{\partial t} + \frac{\partial \psi^0}{\partial x} \frac{\partial \omega^0}{\partial x} - \frac{\partial \psi^0}{\partial y} \frac{\partial \omega^0}{\partial y} &= f & \text{in } \Omega \times (0,T), \\
\psi^0 &= 0 & \text{on } \Gamma, \\
\omega^0|_{t=0} &= \omega_0 & \text{in } \Omega,
\end{align*}
\]

where \( \nabla \perp \psi^0 = \mathbf{u}^0 \) and \( \omega^0 = \text{curl } \mathbf{u}^0 \) for the Euler solution \( \mathbf{u}^0 \).

To fix the discrepancy on the boundary of the Navier-Stokes and Euler vorticities, we propose asymptotic expansions of \( \omega^\varepsilon \) and \( \psi^\varepsilon \),

\[
\omega^\varepsilon \simeq \omega^0 + \Theta, & \quad \psi^\varepsilon \simeq \psi^0 + \theta.
\]

Here \( \Theta \), which balances the difference of \( \omega^\varepsilon - \omega^0 \) on \( \Gamma \), is a vorticity corrector corresponding roughly to the Prandtl equation of the Navier-Stokes vorticity equation. From \( \Theta \), the corrector \( \theta \) of the stream function is defined as well.

Thanks to the estimates on the correctors \( \Theta \) and \( \theta \), performing the energy estimates on the corrected vorticity difference \( \omega^\varepsilon - (\omega^0 + \Theta) \), we improve the convergence results (2.9) as below:
Theorem 2.3. Under some regularity assumptions, the corrected difference \( \psi^\varepsilon - (\psi^0 + \theta) \) of the Navier-Stokes and Euler stream functions converges to zero in the sense that
\[
\| \psi^\varepsilon - (\psi^0 + \theta) \|_{L^\infty(0,T;H^2(\Omega))} \leq \kappa \varepsilon^{\frac{1}{4}}, \quad \| \psi^\varepsilon - (\psi^0 + \theta) \|_{L^2(0,T;H^3(\Omega))} \leq \kappa \varepsilon^{\frac{1}{4}},
\]
for a constant \( \kappa = \kappa(T, \omega_0, f, \Omega) \), independent of \( \varepsilon \). Moreover, we have the uniform \( H^1 \) convergence of the Navier-Stokes solution \( \psi^\varepsilon \) to the Euler solution \( \psi^0 \):
\[
\| \psi^\varepsilon - \psi^0 \|_{L^\infty(0,T;H^1(\Omega))} \leq \kappa \varepsilon^{\frac{1}{4}}.
\]

2.4. Asymptotic expansion of the Stokes solutions at small viscosity: the case of non-compatible initial data. Boundary layers of the Stokes equations (or the linearized Navier-Stokes equations around a tangential flow) are well-studied in, e.g., [TW95, TW96, TW97-1, HT08, 4]. However, in the articles mentioned above, only the case of well-prepared initial data \( u_0 = 0 \) on \( \Gamma \) are considered. As explained in Section 3 of [TW96], to handle the case of ill-prepared initial data, one main difficulty is enforcing the divergence-free constraint on the corrector that satisfies the desired equation and boundary condition. In [HT08] and [4], a divergence-free corrector is introduced while it causes a small error on the boundary. To manage this small error, the authors introduce the supplemented corrector as a solution of the stationary Stokes equations with a constant viscosity. But this approach requires the compatibility condition on the initial data.

In [9], we revisit the Stokes equations, which is (2.1) without the nonlinear term \((\psi^\varepsilon \cdot \nabla)\psi^\varepsilon\), where the initial data \( u_0 \) does not satisfy the compatibility condition. More precisely, we assume that
\[
(2.10) \quad u_0 \in H \cap H^4(\Omega),
\]
where \( H = \{ v \in L^2(\Omega) \mid \text{div } v = 0, \text{ } \vec{v} \cdot \vec{n} = 0 \text{ on } \Gamma \} \). Hence the tangential component of \( u_0 \) does not necessarily vanish on the boundary \( \Gamma \).

Concerning the boundary layer analysis of the Stokes equations at small viscosity, one key observation is that, under the assumption (2.10), the formal limit \( u^0 \), which is a solution of (2.2) without the nonlinear term \((\psi^0 \cdot \nabla)\psi^0\), is sufficiently regular,
\[
\psi^0 \in L^\infty(0,T;H \cap H^4(\Omega)), \quad \frac{\partial \psi^0}{\partial t} \in L^\infty(0,T;H \cap H^4(\Omega)),
\]
provided that \( f \) and \( \Gamma \) are smooth enough.

To the Prandtl equation associated with the Stokes problem in a 3D curved domain, applying the approach in [6], [5] and [TW02], we construct a divergence-free corrector \( \Theta^0 \), and obtain an asymptotic expansion of \( \psi^\varepsilon \) in the form,
\[
\psi^\varepsilon \simeq \psi^0 + \Theta^0.
\]
Here the tangential component of \( \Theta^0 \) is mainly the heat solution in a half-plane.

Under (2.10), some derivatives of \( \Theta^0 \), that appear in the error analysis on \( \psi^\varepsilon - (\psi^0 + \Theta^0) \), have a singularity at \( t = 0 \). However, as this singularity is integrable in \( t \), we obtain the following convergence results:

Theorem 2.4. Under the assumption (2.10), if \( f \) and \( \Gamma \) are sufficiently regular, as the viscosity \( \varepsilon \) tends to zero, the corrected differences of the Stokes solution and the linearized Euler solution converges to zero in the sense that
\[
\| \psi^\varepsilon - (\psi^0 + \Theta^0) \|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \varepsilon^{\frac{1}{4}}, \quad \| \psi^\varepsilon - (\psi^0 + \Theta^0) \|_{L^2(0,T;H^1(\Omega))} \leq \kappa,
\]
for a positive constant $\kappa := \kappa(T, u_0, f, \Omega)$ depending on the data, but independent of $\varepsilon$. Moreover, as $\varepsilon \to 0$, we have the uniform $L^2$ convergence of $u^\varepsilon$ to $u^0$:
\[
\|u^\varepsilon - u^0\|_{L^2((0,T);L^2(\Omega))} \leq \kappa T \varepsilon \frac{1}{\varepsilon}.
\]

2.5. Vorticity layer analysis of some symmetric flows. In [P1], as studied in earlier works [HMMW12, MNW11], we consider the Navier-Stokes equations at small viscosity under some symmetries. More precisely, we consider the Navier-Stokes equations in a periodic channel, $\Omega := \mathbb{R}^2 \times (0, h)$, $h > 0$, and consider plane-parallel solutions,
\[
\mathbf{u}^\varepsilon = (u_1^\varepsilon(z, t), u_2^\varepsilon(x, z, t), 0).
\]

Then the Navier-Stokes equations (2.1) are reduced to the system,
\[
\begin{align*}
\frac{\partial u_1^\varepsilon}{\partial t} - \varepsilon \partial^2 u_1^\varepsilon - \varepsilon \partial^2 u_2^\varepsilon &= f_1, & &\text{in } \Omega \times (0, T), \\
\frac{\partial u_2^\varepsilon}{\partial t} - \varepsilon \partial^2 u_2^\varepsilon &= f_2, & &\text{in } \Omega \times (0, T), \\
\end{align*}
\]
\[
\begin{align*}
\frac{\partial u_1^\varepsilon}{\partial t} &= f_1, & &\text{in } \Omega \times (0, T), \\
\frac{\partial u_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} &= f_2, & &\text{in } \Omega \times (0, T), \\
\end{align*}
\]
\[
\begin{align*}
u_i^\varepsilon &= 0, & &i = 1, 2, \text{ on } \partial \Omega, \\
\left. u_i^\varepsilon \right|_{t=0} &= u_{0,i}, & &i = 1, 2, \text{ in } \Omega.
\end{align*}
\]

Similarly, for the Euler equations (2.2), we consider a plane-parallel solution $\mathbf{u}^0 = (u_1^0(z, t), u_2^0(x, z, t), 0)$, which satisfies
\[
\begin{align*}
\frac{\partial u_1^0}{\partial t} &= f_1, & &\text{in } \Omega \times (0, T), \\
\frac{\partial u_2^0}{\partial t} + u_1^\varepsilon \frac{\partial u_2^0}{\partial x} &= f_2, & &\text{in } \Omega \times (0, T), \\
\end{align*}
\]
\[
\begin{align*}
u_i^0 &= 0, & &i = 1, 2, \text{ on } \partial \Omega, \\
\left. u_i^0 \right|_{t=0} &= u_{0,i}, & &i = 1, 2, \text{ in } \Omega.
\end{align*}
\]

We impose a restriction on the smooth data,
\[
\begin{align*}
f &= 0 \text{ on } \partial \Omega, & &\mathbf{u}_0 = 0 \text{ on } \partial \Omega,
\end{align*}
\]
which implies that $\mathbf{u}^0 = 0$ on $\partial \Omega$. Hence, in this case, by performing the energy estimates on $\mathbf{u}^\varepsilon - \mathbf{u}^0$, one can easily verify the convergence of $\mathbf{u}^\varepsilon$ to $\mathbf{u}^0$ in $L^2$ uniformly in time as well as $L^2$ in time and $H^1$ in space.

The main task of [P1] is, under the assumption (2.12), to obtain the uniform $H^1$ convergence of $\mathbf{u}^\varepsilon$ to $\mathbf{u}^0$. For the plane-parallel flows, as the $H^1$ norm of $\mathbf{u}^\varepsilon$ is equivalent to the $L^2$ norm of curl $\mathbf{u}^\varepsilon$, we consider the vorticity $\omega^\varepsilon := \text{curl } \mathbf{u}^\varepsilon$ which satisfies
\[
\begin{align*}
\frac{\partial \omega_1^\varepsilon}{\partial t} - \varepsilon \partial^2 \omega_1^\varepsilon - \varepsilon \partial^2 \omega_2^\varepsilon - \omega_2^\varepsilon \omega_3^\varepsilon + u_1^\varepsilon \frac{\partial \omega_1^\varepsilon}{\partial z} &= - \frac{\partial f_2}{\partial z}, & &\text{in } \Omega \times (0, T), \\
\frac{\partial \omega_2^\varepsilon}{\partial t} - \varepsilon \partial^2 \omega_2^\varepsilon - \omega_2^\varepsilon \omega_3^\varepsilon + u_1^\varepsilon \frac{\partial \omega_2^\varepsilon}{\partial x} &= - \frac{\partial f_1}{\partial x}, & &\text{in } \Omega \times (0, T), \\
\frac{\partial \omega_3^\varepsilon}{\partial t} - \varepsilon \partial^2 \omega_3^\varepsilon - \omega_2^\varepsilon \omega_3^\varepsilon + u_1^\varepsilon \frac{\partial \omega_3^\varepsilon}{\partial x} &= - \frac{\partial f_2}{\partial x}, & &\text{in } \Omega \times (0, T), \\
\frac{\partial \omega_1^\varepsilon}{\partial z} &= \frac{\partial \omega_2^\varepsilon}{\partial z} = \omega_3^\varepsilon = 0, & &\text{on } \partial \Omega,
\end{align*}
\]
\[
\left. \omega^\varepsilon \right|_{t=0} = \omega_0 = \text{curl } \mathbf{u}_0.
\]
Taking the curl of (2.11), we find the corresponding limit problem for \( \omega^0 := \text{curl } u^0 \),

\[
\begin{aligned}
\frac{\partial \omega^0_1}{\partial t} - \omega^0_2 \omega^0_3 + u^0_1 \frac{\partial \omega^0_1}{\partial x} &= - \frac{\partial f^2}{\partial z}, & \text{in } \Omega \times (0, T), \\
\frac{\partial \omega^0_2}{\partial t} &= \frac{\partial f_1}{\partial z}, & \text{in } \Omega \times (0, T), \\
\frac{\partial \omega^0_3}{\partial t} + u^0_1 \frac{\partial \omega^0_3}{\partial x} &= \frac{\partial f_2}{\partial x}, & \text{in } \Omega \times (0, T), \\
\omega^0_3 &= 0, & \text{on } \partial \Omega, \\
\omega^0|_{t=0} &= \omega_0.
\end{aligned}
\]

Comparing the boundary conditions, we see that, on the boundary, the tangential components of \( \frac{\partial \omega^\varepsilon}{\partial z} \) and \( \frac{\partial \omega^0}{\partial z} \) do not match, hence we expect that the (weak) boundary layers of the vorticity to appear. To manage this difficulty, we propose an asymptotic expansion of \( \omega^\varepsilon \) in the form,

\[
\omega^\varepsilon \simeq \omega^0 + \theta,
\]

where \( \theta = (\theta_1(z, t), \theta_2(x, z, t), 0) \), and each \( \theta_i, i = 1, 2 \), is defined mainly as a solution to the heat equation in a half plane, supplemented with Neumann boundary condition.

Performing the error analysis, we obtain the convergence results of the vorticity:

**Theorem 2.5.** *Assuming that the data \( f \) and \( u_0 \), satisfying (2.12), are sufficiently regular, \( \omega^\varepsilon \) converges to \( \omega^0 \) as \( \varepsilon \to 0 \) in the sense that*

\[
\begin{aligned}
\| \omega^\varepsilon_1 - \omega^0_1 \|_{L^\infty(0, T; L^2(\Omega))} &\leq \kappa \varepsilon^{\frac{3}{4}}, \\
\| \omega^\varepsilon_2 - \omega^0_2 \|_{L^2(0, T; H^1(\Omega))} &\leq \kappa \varepsilon^{\frac{1}{4}}, \\
\| \omega^\varepsilon_3 - \omega^0_3 \|_{L^\infty(0, T; L^2(\Omega))} &\leq \kappa \varepsilon, \\
\| \omega^\varepsilon_3 - \omega^0_3 \|_{L^2(0, T; H^1(\Omega))} &\leq \kappa \varepsilon^{\frac{1}{2}},
\end{aligned}
\]

*for a constant \( \kappa = \kappa(\Omega, f, u_0, T) \), independent of \( \varepsilon \).*

### 3. Current and Future Projects

3.1. **Vorticity layer analysis of some symmetric flows: The case of non compatible initial data.** With J. P. Kelliher, M. C. Lopes Filho, A. Mazzucato and H. J. Nussenzveig Lopes, we currently study the boundary layers of the plane-parallel or infinite-pipe flows. Here, by dropping the condition (2.12), we do not impose any compatibility condition on the smooth data. Hence, comparing to some earlier results, e.g, [HMNW12, MNW11, P1], we expect to obtain much weaker convergence results of the Navier-Stokes to the Euler solution, but similar to those appearing in [LMNT08].

3.2. **Techniques on the boundary layer analysis.** With M. Hamouda, C.-Y. Jung and R. Temam, in [P4, P2], we study the boundary layers of some convection-diffusion equations in a 2D rectangular domain. The main goal of this project is to understand and resolve some technical difficulties on the boundary layer analysis. We believe that the methods introduced in this project can be made useful in many applications, including the Navier-Stokes equations.

In [P4], we consider the case when the limit solution has an interior shock which is not perpendicular to the ordinary boundary layer of the problem. In [P2], the singular perturbation problem with an ordinary and parabolic boundary layers is considered, when two types of boundary layers intersect near a corner of the rectangular domain. Each
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of interior, ordinary and parabolic boundary layers can be well-treated by introducing a corrector, and now we are currently constructing a corrector which estimating the interactions of the layers.

3.3. Cell-centered Finite Volume method for a class of elliptic problems. As an application of [2], with C.-Y. Jung and T.-B. Nguyen, in [P3], we study the cell-centered Finite Volume discretization of some elliptic problems in a 2D rectangular domain. Here we follow and modify the approach in Section 7 of [2] so that the numerical scheme for simulations is obtained simply by setting the test function as one in the discretized variational problem. We believe that this new numerical scheme is better than the classical cell centered Finite Volume scheme because, for the analysis, any symmetry on the elliptic differential operator is not required. The convergence of the Finite Volume approximate solution to the weak solution is proved in [P3], and we are currently performing some related numerical simulations to support the theoretical results.

LIST OF PUBLICATIONS

[8] (with C.-Y. Jung), Vorticity layers of the 2D Navier-Stokes equations with a slip type boundary condition, Submitted 2, 5
[6] (with M. Hamouda and R. Temam), Asymptotic analysis of the Navier-Stokes equations in a curved domain with a non-characteristic boundary, To appear in Networks and Heterogeneous Media, Special Issue in honor of Hiroshi Matano 2, 3, 5, 6

WORK IN PROGRESS

[P1] Vorticity layer analysis of some symmetric flows, Preprint 2, 7, 8
[P2] (with C.-Y. Jung and R. Temam), Singular Perturbation Analysis for convection-diffusion equations with corners, Preprint 8
[P3] (with C.-Y. Jung and T.-B. Nguyen), Cell-centered Finite Volume method for a class of elliptic problems, Preprint 9
[P4] (with M. Hamouda and C.-Y. Jung), Interaction of interior characteristic layers and boundary layers 8
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