Lie *n*-algebras, supersymmetry, and division algebras

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Higher Structures IV

This research began as a puzzle. Explain this pattern:

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They have dimensions $k = 1, 2, 4$ and 8.
- The classical superstring makes sense only in dimensions $k + 2 = 3, 4, 6$ and 10.
- The classical super-2-brane makes sense only in dimensions $k + 3 = 4, 5, 7$ and 11.

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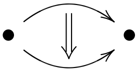
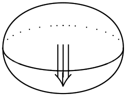
Pulling on this thread will lead us into higher gauge theory.

Higher Gauge Theory			
Object	Parallel transport	Holonomy	Infinitesimally
Particle		Lie group	Lie algebra
String		Lie 2-group	Lie 2-algebra
2-Brane		Lie 3-group	Lie 3-algebra

- Everything in this table can be made "super".
- A connection valued in Lie *n*-algebra is a connection on an *n*-bundle, which is like a bundle, but the fibers are "smooth *n*-categories."
- The theory of Lie *n*-algebra-valued connections was developed by Hisham Sati, Jim Stasheff and Urs Schreiber.
- Let us denote the Lie 2-superalgebra for superstrings by $\mathfrak{superstring}$.
- Let us denote the Lie 3-superalgebra for 2-branes by $2\text{-}\mathfrak{brane}$.

- Yet superstrings and super-2-branes are *exceptional objects*—they only make sense in certain dimensions.
- The corresponding Lie 2- and Lie 3-superalgebras are similarly exceptional.
- Like many exceptional objects in mathematics, they are tied to the division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.
- In this talk, I will show you how $\mathfrak{superstring}$ and 2-$\mathfrak{brane}$ arise from division algebras.

But why should we care about $\mathfrak{superstring}$ and 2-$\mathfrak{brane}$?

- In dimensions 3, 4, 6 and 10, we will define the superstring Lie 2-superalgebra to be the chain complex:

$$\mathfrak{siso}(V) \leftarrow \mathbb{R}$$

  This is Lie 2-superalgebra extending the Poincaré Lie superalgebra, $\mathfrak{siso}(V)$.

- In dimensions 4, 5, 7 and 11, we will define the 2-brane Lie 3-superalgebra to be a chain complex:

$$\mathfrak{siso}(\mathcal{V}) \leftarrow 0 \leftarrow \mathbb{R}$$

  This is a Lie 3-superalgebra extending the Poincaré Lie superalgebra, $\mathfrak{siso}(\mathcal{V})$.

Connections valued in these Lie *n*-superalgebras describe the *parallel transport* of superstrings and super-2-branes in the appropriate dimension:

$\mathfrak{superstring}(V)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 2-form, the B field.
$\downarrow$	
$\mathfrak{siso}(V)$	$\mathfrak{siso}(V)$-valued 1-form.

2-$\mathfrak{brane}(\mathcal{V})$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 3-form, the C field.
$\downarrow$	
0	
$\downarrow$	
$\mathfrak{siso}(\mathcal{V})$	$\mathfrak{siso}(\mathcal{V})$-valued 1-form.

The B and C fields are very important in physics…

- The B field, or Kalb-Ramond field, is to the string what the electromagnetic A field is to the particle.
- The C field is to the 2-brane what the electromagnetic A field is to the particle.

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…and geometry:

- The A field is really a connection on a $\mathrm{U}(1)$-bundle.
- The B field is really a connection on a $\mathrm{U}(1)$-gerbe, or 2-bundle.
- The C field is really a connection on a $\mathrm{U}(1)$-2-gerbe, or 3-bundle.

Using $\mathfrak{superstring}$ and 2-$\mathfrak{brane}$, we neatly package these fields with the Levi–Civita connection on spacetime.

Let us see where these Lie *n*-superalgebras come from, starting with the reason superstrings and 2-branes only make sense in certain dimensions.

In the physics literature, the classical superstring and super-2-brane require certain spinor identities to hold:

Superstring In dimensions 3, 4, 6 and 10, we have:

$$[\psi, \psi]\psi = 0$$

for all spinors $\psi \in S$.

Here, we have:

- the bracket is a symmetric map from spinors to vectors:

$$[,]\colon \mathrm{Sym}^2 S \to V$$

- vectors can "act" on spinors via the Clifford action, since $V \subseteq \mathrm{Cliff}(V)$.

Recall that:

- V is the vector representation of $\mathrm{Spin}(V) = \widetilde{\mathrm{SO}_0(V)}$.
- S is a spinor representation, i.e. a representation coming from a module of $\mathrm{Cliff}(V)$.
- $\mathrm{Cliff}(V) = \frac{TV}{v^2 = ||v||^2}$.

Similarly, for the 2-brane:

Super-2-brane In dimensions 4, 5, 7 and 11, the 3-ψ's rule need not hold:

$$[\Psi, \Psi]\Psi \neq 0$$

Instead, we have the 4-Ψ's rule:

$$[\Psi, [\Psi, \Psi]\Psi] = 0$$

for all spinors $\Psi \in \mathcal{S}$.

Again:

- $\mathcal{V}$ and $\mathcal{S}$ are vectors and spinors for these dimensions.
- $[,] \colon \mathrm{Sym}^2\mathcal{S} \to \mathcal{V}$.

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- We can use $\mathbb{K}$ to build V and S in dimensions 3, 4, 6 and 10, $\mathcal{V}$ and $\mathcal{S}$ in 4, 5, 7 and 11.
- The 3-ψ's and 4-Ψ's rules are consequences of this construction.

In superstring dimensions 3, 4, 6 and 10:

- The vectors V are the 2×2 Hermitian matrices with entries in $\mathbb{K}$:

$$V = \left\{ \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} : t, x \in \mathbb{R}, \quad y \in \mathbb{K} \right\}.$$

- The determinant is then the norm:

$$-\det \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} = -t^2 + x^2 + |y|^2.$$

- This uses the properties of $\mathbb{K}$:

$$|y|^2 = y\bar{y}.$$

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- Showing

$$[\psi, \psi]\psi = 0$$

  is now an easy calculation!

- These constructions are originally due to Tony Sudbery, with help from Corrinne Manogue, Tevian Dray and Jorg Schray.
- We have shown to generalize them to the 2-brane dimensions 4, 5, 7 and 11, taking $\mathcal{V} \subseteq \mathbb{K}[4]$ and $\mathcal{S} = \mathbb{K}^4$.
- The 4-Ψ's rule

$$[\Psi, [\Psi, \Psi]\Psi] = 0$$

is then also an easy calculation.

What are the 3-ψ's and 4-Ψ's rules?
They are *cocycle conditions*.

- In 3, 4, 6 and 10, there is a 3-cochain α:

$$\alpha(\psi, \phi, v) = \langle \psi, v\phi \rangle.$$

  Here, $\langle -, - \rangle$ is a $\mathrm{Spin}(V)$-invariant pairing on spinors.
- $d\alpha = 0$ is the 3-ψ's rule!

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- $d\alpha = 0$ is the 3-ψ's rule!
- In 4, 5, 7 and 11, there is a 4-cochain β:

$$\beta(\Psi, \Phi, V, W) = \langle \Psi, (VW - WV)\Phi \rangle.$$

  Here, $\langle -, - \rangle$ is a $\mathrm{Spin}(\mathcal{V})$-invariant pairing on spinors.
- $d\beta = 0$ is the 4-Ψ's rule!

Lie (super)algebra cohomology:

- Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra,
- which has bracket $[,] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}$,
- where $\Lambda^2 \mathfrak{g} = \Lambda^2 \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_1 \oplus \mathrm{Sym}^2 \mathfrak{g}_1$ is the graded exterior square.
- We get a cochain complex:

$$\Lambda^0 \mathfrak{g}^* \to \Lambda^1 \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \to \cdots$$

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$$\Lambda^0\mathfrak{g}^* \to \Lambda^1\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^* \to \cdots$$

- where $d = [,]^*\colon \Lambda^1\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$, the dual of the bracket.
- $d^2 = 0$ is the Jacobi identity!

- In 3, 4, 6 and 10:
$$T = V \oplus S$$
is a Lie superalgebra, with bracket
$$[,] \colon \mathrm{Sym}^2 S \to V.$$
- $\alpha(\psi, \phi, v) = \langle \psi, v\phi \rangle$ is a 3-cocycle on T.
- In 4, 5, 7 and 11:
$$\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$$
is a Lie superalgebra, with bracket
$$[,] \colon \mathrm{Sym}^2 \mathcal{S} \to \mathcal{V}.$$
- $\beta(\Psi, \Phi, V, W) = \langle \Psi, (VW - WV)\Phi \rangle$ is a 4-cocycle on $\mathcal{T}$.

- In 3, 4, 6 and 10: we can extend α to a cocycle on

$$\mathfrak{siso}(V) = \mathfrak{spin}(V) \ltimes T$$

the Poincaré superalgebra.

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- In 4, 5, 7 and 11: we can extend β to a cocycle on

$$\mathfrak{siso}(\mathcal{V}) = \mathfrak{spin}(\mathcal{V}) \ltimes \mathcal{T}$$

  the Poincaré superalgebra.

The spinor identities were cocycle conditions for α and β. What are α and β good for?

Building Lie *n*-superalgebras!

Definition

A **Lie *n*-superalgebra** is an *n* term chain complex of $\mathbb{Z}_2$-graded vector spaces:

$$L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_{n-1}$$

endowed with a bracket that satisfies Lie superalgebra axioms up to chain homotopy.

This is a special case of an L_∞-superalgebra.

Definition

An $\mathbf{L_\infty}$**-algebra** is a graded vector space L equipped with a system of grade-antisymmetric linear maps

$$[-, \cdots, -] \colon L^{\otimes k} \to L$$

satisfying a generalization of the Jacobi identity.

So L has:

- a boundary operator $\partial = [-]$ making it a chain complex,
- a bilinear bracket $[-, -]$, like a Lie algebra,
- but also a trilinear bracket $[-, -, -]$ and higher, all satisfying various identities.

The following theorem says we can package cocycles into Lie *n*-superalgebras:

Theorem (Baez–Crans)

If ω *is an* $n+1$ *cocycle on the Lie superalgebra* $\mathfrak{g}$*, then the* n *term chain complex*

$$\mathfrak{g} \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow \mathbb{R}$$

equipped with

$$[-,-] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}$$

$$\omega = [-, \cdots, -] \colon \Lambda^{n+1} \mathfrak{g} \to \mathbb{R}$$

is a Lie n-superalgebra.

Theorem

In dimensions 3, 4, 6 and 10, there exists a Lie 2-superalgebra, which we call $\mathfrak{superstring}(\boldsymbol{V})$, *formed by extending the Poincaré superalgebra* $\mathfrak{siso}(V)$ *by the 3-cocycle* α.

Theorem

In dimensions 4, 5, 7 and 11, there exists a Lie 3-superalgebra, which we call $\mathbf{2}\text{-}\mathfrak{brane}(\boldsymbol{\mathcal{V}})$, *formed by extending the Poincaré superalgebra* $\mathfrak{siso}(\mathcal{V})$ *by the 4-cocycle* β.