WEYL GROUPS

Let's compute Weyl groups for:

A_n	$SU(n+1)$	$W = S_{n+1}$
B_n	$Spin(2n+1)$	$W = S_n \ltimes \mathbb{Z}_2^n$
D_n	$Spin(2n)$	$W = S_n \ltimes \mathbb{Z}_2^{n-1}$

Here S_n acts on $\mathbb{Z}_2^n$ by permuting coordinates. Think of $\mathbb{Z}_2^{n-1} \subseteq \mathbb{Z}_2^n$ as the subgroup $\{(x_1, \ldots, x_n) : \sum x_i = 0\}$; the action of S_n on $\mathbb{Z}_2^n$ preserves this subgroup.

For A_n:

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_{n+1} \end{pmatrix} : \theta_i \in \mathbb{R} \text{ s.t. } \sum \theta_i = 0 \right\}$$

Here $W = N(T)/T$; typical elts of $N(T)$ include:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SU(3)$$

or more generally

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix}$$

which acts by conjugation on $\mathfrak{t}$.

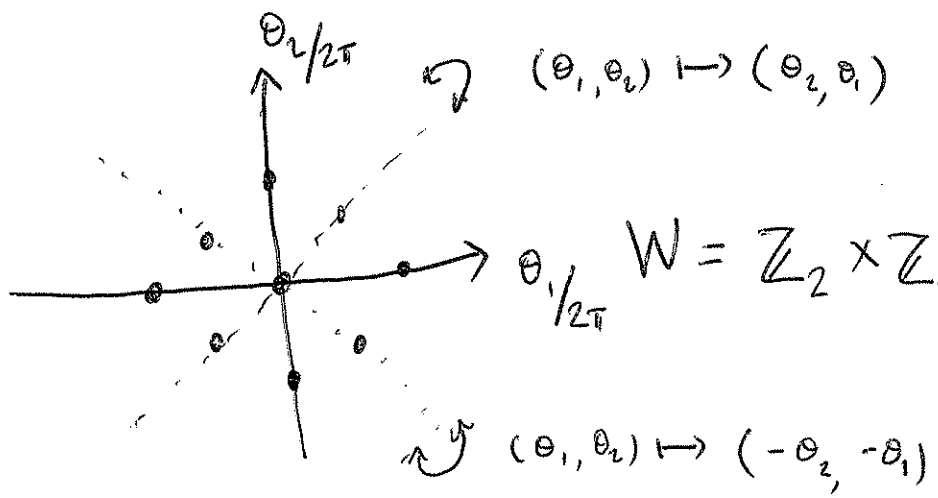
$$(\theta_1, \dots, \theta_{n+1}) \longmapsto (\theta_1, \dots, \theta_{k+1}, \theta_k, \dots, \theta_{n+1})$$

This is an elementary transposition in S_{n+1}; these generate S_{n+1} so W is at least S_{n+1}.

In fact, $W \cong S_{n+1}$.

Next: D_n (or $\mathrm{Spin}(2n)$). Recall our work on $\mathrm{Spin}(4)$:

$L \subseteq \mathfrak{t}$



$W = \mathbb{Z}_2 \times \mathbb{Z}_2$

In general

$$T = \left\{ \begin{pmatrix} e^{J\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{J\theta_n} \end{pmatrix} : \theta_i \in \mathbb{R} \right\}_{\text{double cover}} \subseteq Spin(2n)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e^{J\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

For $Spin(4)$, we have two generators of W:

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \text{where} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \quad \text{where } K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$K^2 = \mathbb{1}, \quad KJK^{-1} = -J,$$
$$JK = -KJ, \quad Ke^{\theta J}K^{-1} = e^{-\theta J}$$

In effect

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\theta_1 J} & 0 \\ 0 & e^{\theta_2 J} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{\theta_2 J} & 0 \\ 0 & e^{\theta_1 J} \end{pmatrix}$$

i.e. $(\theta_1, \theta_2) \longmapsto (\theta_2, \theta_1)$

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} e^{\theta_1 J} & 0 \\ 0 & e^{\theta_2 J} \end{pmatrix} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} K e^{\theta_2 J} K & 0 \\ 0 & K e^{\theta_1 J} K \end{pmatrix} = \begin{pmatrix} e^{-\theta_2 J} & 0 \\ 0 & e^{-\theta_1 J} \end{pmatrix}$$

I.e. $(\theta_1, \theta_2) \longmapsto (-\theta_2, -\theta_1)$

Now let's generalize to $Spin(2n)$:

E.g. $n=3$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ acting on $(\theta_1, \theta_2, \theta_3) \in t$

by $(\theta_1, \theta_2, \theta_3) \longmapsto (\theta_2, \theta_1, \theta_3)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (\theta_1, \theta_2, \theta_3) \longmapsto (\theta_1, \theta_3, \theta_2)$$

$$\begin{pmatrix} 0 & K & 0 \\ -K & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\theta_1, \theta_2, \theta_3) \longmapsto (-\theta_2, -\theta_1, \theta_3)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & K \\ 0 & -K & 0 \end{pmatrix} \quad (\theta_1, \theta_2, \theta_3) \longmapsto (\theta_1, -\theta_3, -\theta_2)$$

This kind of elt generates all of W, so W acts on $\mathfrak{t} \cong \mathbb{R}^n$ as the following transformations:

$$(\theta_1, \theta_2, \ldots, \theta_n) \longmapsto (\pm\theta_{\sigma(1)}, \ldots, \pm\theta_{\sigma(n)})$$

where $\sigma \in S_n$ & there are an <u>even</u> number of minus signs. So

$$W = S_n \ltimes \mathbb{Z}_2^{n-1}$$

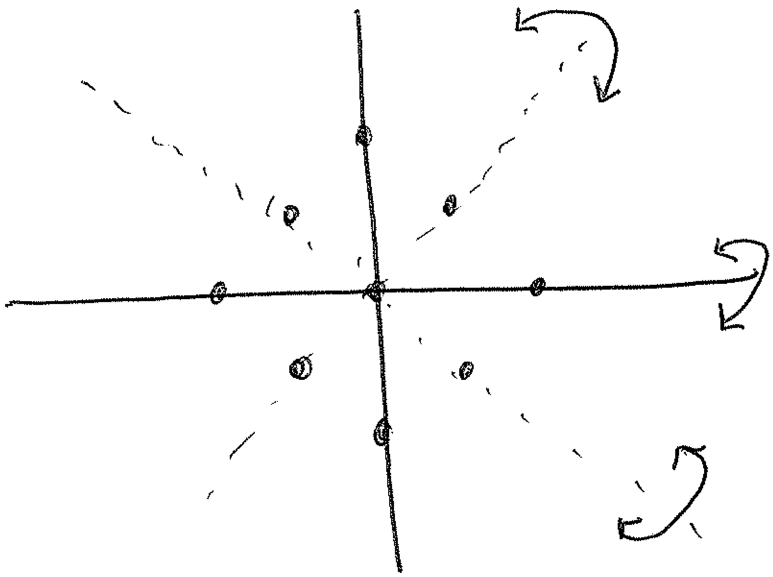
Next: B_n (i.e. $Spin(2n+1)$)

Here

$$T = \left\{ \begin{pmatrix} e^{\theta_1 J} & & 0 \\ & \ddots & \\ 0 & & e^{\theta_n J} \\ & & & 1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$$

Same as for $Spin(2n)$!

So $\mathfrak{t}$ & L are the same for B_n as D_n, but W is bigger!



$(\theta_1, \theta_2) \mapsto (\theta_1, -\theta_2)$

One extra reflection generating W.

This reflection comes from:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e^{\theta_1 J} & 0 & 0 \\ 0 & e^{\theta_2 J} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{\theta_1 J} & 0 & 0 \\ 0 & e^{-\theta_2 J} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get generators for W of $Spin(2n+1)$ in the same way, using:

$$\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & -1 \end{pmatrix}$$

which has the effect:

$$(\theta_1, \dots, \theta_n) \longmapsto (\theta_1, \dots, -\theta_n)$$

So W includes ($\cong$ equals)

$$S_n \ltimes \mathbb{Z}_2^n$$

What's all this good for? One thing: You can understand the set of conjugacy classes in our group K.

Every elt of K is conjugate to one in $T \subseteq K$.

So each conjugacy class has a representative in T, & two elements $t, t' \in T$ lie in the same conjugacy class iff $\exists w \in W$ s.t. $t = wt'w^{-1}$.

So:

$$\{\text{conjugacy classes in } K\} \cong \{W\text{-orbits in } T\}$$