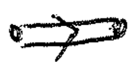
Lie group coincidences

	B_1	$\mathrm{Spin}(3) \cong SU(2)$	A_1	
	D_2	$\mathrm{Spin}(4) \cong SU(2)\times SU(2)$	$A_1 \times A_1$	
	B_2	$\mathrm{Spin}(5) \cong Sp(2)$	C_2	
	D_3	$\mathrm{Spin}(6) \cong SU(4)$	A_3	

Last time, we saw $SU(4)$ double covers $SO(6)$, so $SU(4) \cong \mathrm{Spin}(6)$. Now let's use this to show $Sp(2)$ double covers $SO(5)$, & so $Sp(2) \cong \mathrm{Spin}(5)$.

We'll see that $Sp(2)$ is a subgroup of $SU(4)$, & obviously $SO(5)$ is a subgroup of $SO(6)$, & indeed we have:

$$\begin{tikzcd} Sp(2) \arrow[r, "2-1"] \arrow[d] & SO(5) \arrow[d] \\ SU(4) \arrow[r, "2-1"] & SO(6) \end{tikzcd}$$

In fact, $Sp(n)$ is a subgroup of $SU(2n)$.

Why? $Sp(n)$ is all $\mathbb{H}$-linear transformations of $\mathbb{H}^n$ preserving the quaternion-valued inner product:

$$(v, w) = \sum_{i=1}^{n} v_i \overline{w_i}, \qquad v_i, w_i \in \mathbb{H}$$

$U(2n)$ is all $\mathbb{C}$-linear transformations of $\mathbb{C}^{2n}$ preserving the complex inner product:

$$\langle v, w \rangle = \sum_{i=1}^{2n} v_i \overline{w_i}, \qquad v_i, w_i \in \mathbb{C}$$

To think of $\mathbb{C}^{2n}$ as being $\mathbb{H}^n$, we need to equip it w/ extra structure, & the subgroup of $U(2n)$ preserving this extra structure is $Sp(n)$.

In fact, this $Sp(n)$ lies in $SU(2n)$, but we won't prove that.

To make a $2n$-dimensional complex vector space V into an n-dimensional quaternionic vector space, we need to equip it with an operator

$$j : V \longrightarrow V$$

with $j^2 = -1$, & $ij = -ji$. (So j is a real-linear but not complex-linear operator. Instead, it's <u>conjugate linear</u>!)

We then define $k = ij$, & check:

$$k^2 = -1$$

$$ij = k = -ji \quad \text{\& cyclic}$$

$$ijk = -1.$$

So a <u>quaternionic</u> <u>str</u> on a complex vector space is a conjugate-linear j w/ $j^2 = -1$.

If our complex vector space had an inner product $\langle \cdot, \cdot \rangle$ on it, and we're trying to make it into a quaternionic inner product space w/ inner product $(\cdot, \cdot)$, then we also need

$$\langle jv, jw \rangle = \overline{\langle v, w \rangle}$$

Then we can define:

$$(v, w) = \langle v, w \rangle - j \langle jv, w \rangle \in \mathbb{H}$$

(or something like it!), and we get a quaternionic inner product space. This needs to be checked!

Back to showing

$$\begin{array}{ccc} SU(4) & \longrightarrow & SO(6) \\ \cup & & \cup \\ Sp(2) & \longrightarrow & SO(5) \end{array}$$

Last time we got the double cover $SU(4) \to SO(6)$ as follows. We saw $SU(4)$ acts on $\mathbb{C}^4$ & thus on the exterior algebra $\Lambda \mathbb{C}^4$, preserving the Hodge star operator:

$$* : \Lambda^p \mathbb{C}^4 \longrightarrow \Lambda^{4-p} \mathbb{C}^4$$

s.t.

$$\omega \wedge * \nu = \langle \omega, \nu \rangle \, \mathrm{vol}$$

where $\mathrm{vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$, & $\langle \cdot, \cdot \rangle$ on $\Lambda^p \mathbb{C}^4$ comes from $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^4$.

We saw that $\Lambda^2 \mathbb{C}^4$ is $\binom{4}{2} = 6$ dimensional complex inner product space, on which $SU(4)$ acts.

We also saw that $* : \Lambda^2 \mathbb{C}^4 \to \Lambda^2 \mathbb{C}^4$ is conjugate-linear & $*^2 = 1$.

A conjugate-linear operator $* : V \to V$ w/ $*^2 = 1$ is called a <u>real structure</u>, since it allows us to define real vector spaces

$$\mathrm{Re}\, V = \{ v \in V : * v = v \}$$

$$\mathrm{Im}\, V = \{ v \in V : * v = -v \}$$

s.t.

$$\mathrm{Re}\, V \oplus \mathrm{Im}\, V \cong V$$

as real vector spaces, &

$$\begin{tikzcd} \mathrm{Re}\, V \arrow[r, bend left, "i"] & \mathrm{Im}(V) \arrow[l, bend left, "i"] \end{tikzcd}$$

so

$$V \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Re}\, V$$

Since $SU(4)$ acts on $\Lambda^2 \mathbb{C}^4$ preserving $*$, it acts on the subspace $\mathrm{Re}(\Lambda^2\mathbb{C}^4)$, which is a 6-dim. real vector space. We saw this gives a homomorphism

$$SU(4) \longrightarrow SO(6)$$

which is 2-1 & onto.

Now we want $Sp(2) \longrightarrow SO(5)$.

For this, take $\mathbb{C}^4$ & equip it w/ $j: \mathbb{C}^4 \to \mathbb{C}^4$ making it into a quaternionic inner product space. $Sp(2)$ will be the subgroup of of $SU(4)$ preserving this extra structure.

So $Sp(2)$ also acts on $\mathrm{Re}(\Lambda^2\mathbb{C}^4)$, so we get:

$$\begin{tikzcd}
Sp(2) \arrow[rr] \arrow[dr, hook] & & SO(6) \\
& SU(4) \arrow[ur] &
\end{tikzcd}$$

In fact, the image of $Sp(2)$ lies in $SO(5)$. Why?

It's because $\Lambda^2\mathbb{C}^4$ gets a special element in it, coming from $j: \mathbb{C}^4 \to \mathbb{C}^4$. $Sp(2)$ preserves this special element $\omega \in \Lambda^2\mathbb{C}^4$, so it preserves the 5-dim. complex inner product space $\omega^{\perp} \subseteq \Lambda^2\mathbb{C}^4$. In fact, $\omega \in \mathrm{Re}(\Lambda^2\mathbb{C}^4)$, so $Sp(2)$ preserves the 5-dim real space

$\omega^\perp \subseteq Re(\Lambda^2 \mathbb{C}^4)$.

This gives a homomorphism

$$Sp(2) \longrightarrow SO(5)$$

which is 2-1 & onto.

To see this, we need to check:

1. $\mathfrak{sp}(2) \longrightarrow \mathfrak{so}(5)$ is 1-1.

2. Therefore its onto if the dimensions agree;

$$\dim \mathfrak{so}(5) = 1+2+3+4 = 10$$

$$\dim \mathfrak{sp}(2) = 3+3+4 = 10$$

3. Thus $Sp(2) \longrightarrow SO(5)$ is
defined if $Sp(2)$ is simply-connected. ✓
Onto if $SO(5)$ is connected. ✓

ξ thus has discrete kernel, which turns out to be $\mathbb{Z}_2$.

How do we get $\omega \in \Lambda^2 \mathbb{C}^4$? We get it from an elt of $\Lambda^2 \mathbb{C}^{4*}$, which we can identify w/ $\Lambda^2 \mathbb{C}^4$. An elt of $\Lambda^2 \mathbb{C}^{4*}$ is a skew symmetric bilinear map

$$\omega : \mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C}$$

We build this from the inner product:

$$\langle \cdot, \cdot \rangle : \mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C}$$

Which alas is conjugate - linear in the second slot, ξ

$$j : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$$

Which is conjugate-linear, so:

$$\omega(v, w) = \langle v, jw \rangle$$

This is bilinear, and also skew-symmetric:

$$\begin{aligned} \omega(v, w) &= \langle v, jw \rangle \\ &= \overline{\langle jv, j^2 w \rangle} \\ &= -\overline{\langle jv, w \rangle} \\ &= -\langle w, jv \rangle \\ &= -\omega(v, w) \end{aligned}$$