The Quaternions

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- The complex numbers $\mathbb{C}$ form a plane.
- Their operations are very related to two dimensional geometry.
- In particular, multiplication by a unit complex number:

$$|z|^2 = 1$$

gives a *rotation*:

$$R_z(w) = zw.$$

Remember:

- $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, \quad i^2 = -1\}$
- We can add and multiply in $\mathbb{C}$.
- We can also conjugate:

$$\overline{a + bi} = a - bi.$$

- And take norms:

$$|z|^2 = z\bar{z}$$

  where

$$|a + bi|^2 = a^2 + b^2.$$

- This norm is crucial! It satisfies:

$$|zw| = |z||w|.$$

The unit complex numbers

$$\mathrm{U}(1) = \left\{ z \in \mathbb{C} : |z|^2 = 1 \right\}$$

form a circle. They also form a group:

- $\mathrm{U}(1)$ is closed under multiplication:

$$|zw| = |z||w| = 1$$

- The conjugates are inverses:

$$|\overline{z}| = |z| = 1$$

and

$$z\overline{z} = |z|^2 = 1$$

.

The key group in plane geometry is $\mathrm{SO}(2)$, the group of rotations of the plane.

- We have a map

$$\begin{array}{cccc} \varphi : & \mathrm{U}(1) & \longrightarrow & \mathrm{SO}(2) \\ & z & \longmapsto & R_z \end{array}$$

- This map is an isomorphism!

$$\mathrm{U}(1) \cong \mathrm{SO}(2).$$

So these groups are the same.

The 19th century Irish mathematician and physicist William Rowan Hamilton was fascinated by the role of $\mathbb{C}$ in two-dimensional geometry.

For years, he tried to invent an algebra of "triplets" to play the same role in three dimenions:

$$a + bi + cj \in \mathbb{R}^3.$$

Alas, we now know this quest was in vain.

Theorem

The only normed division algebras, which have a norm satisfying

$$|zw| = |z||w|$$

have dimension 1, 2, 4, or 8.

Hamilton's search continued into October, 1843:

Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: "Well, Papa, can you multiply triplets?" Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

On October 16th, 1843, while walking with his wife in to a meeting of the Royal Society of Dublin, Hamilton discovered a 4-dimensional algebra called the **quaternions**:

That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since:

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternions are

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

- $ij = k = -ji$.
- We can conjugate them:

$$\overline{a + bi + cj + dk} = a - bi - cj - dk.$$

- There's a norm:

$$|q|^2 = q\overline{q}.$$

- Satisfying:

$$|qu| = |q||u|$$

Just like the unit complex numbers, the unit quaternions form a group:

$$\mathrm{Spin}(3) = \left\{ q \in \mathbb{H} : |q|^2 = 1 \right\}$$

We can use these to give rotations in three dimensions!

- Think of $\mathbb{R}^3$ as the imaginary quaternions:

$$\mathrm{Im}\,\mathbb{H} = \{ai + bj + ck : a, b, c \in \mathbb{R}\}$$

- A unit quaternion gives a rotation:

$$R_q(\mathbf{v}) = q\mathbf{v}\overline{q}, \quad \mathbf{v} \in \mathrm{Im}\,\mathbb{H}.$$

- We can represent any rotation this way.
- This is often used in programming applications.
- We get a map:

$$\begin{array}{rccc} \varphi : & \mathrm{Spin}(3) & \longrightarrow & \mathrm{SO}(3) \\ & q & \longmapsto & R_q \end{array}$$

where $\mathrm{SO}(3)$ is the group of rotations in three-dimensions.

Is this map an isomorphism?

No!

It's onto, but not 1-to-1:

$$R_q = R_{-q}, \text{ since } q\mathbf{v}\overline{q} = (-q)\mathbf{v}(-\overline{q})$$

In fact, it's 2-to-1.

So, φ is not an isomorphism. Is there an isomorphism?

$$\mathrm{Spin}(3) \cong \mathrm{SO}(3)?$$

We can answer this using *topology*.

- As a space:

$$\begin{aligned}\mathrm{Spin}(3) &= \left\{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\right\} \\ &= S^3.\end{aligned}$$

  the 3-sphere, a sphere of higher dimension.
- $\mathrm{SO}(3) =$ ball of radius π with antipodal points identified.

These spaces sound different. *Are they?*

Yes!

- $\mathrm{Spin}(3) = S^3$ is **simply connected:** any loop in it can be continuously deformed to a point.
- $\mathrm{SO}(3)$ is not simply connected: there is a loop that can't be deformed to a point.

Therefore:

$$\mathrm{Spin}(3) \not\cong \mathrm{SO}(3).$$

Amazingly, this fact is important in quantum physics!

- A path from 0 to 360 in $\mathrm{Spin}(3)$ starts at 1, but ends at -1.
- Since $R_1 = R_{-1}$, this is OK.
- You must rotate from 0 to 720 to get back to 1.

Quantum mechanics says that particles are represented by waves:

- The simplest kind of wave a is a function:

$$\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$$

- But since we live in three dimensions:

$$\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$$

- And because it's quantum:

$$\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{C}$$

- Yet, in 1924, Wolfgang Pauli (secretly) discovered that for electrons:

$$\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{H}$$

If we rotate most particles, we rotate its wave:

$$R_q\psi(\mathbf{v}) := \psi(R_{\overline{q}}(\mathbf{v})).$$

But to rotate an electron, Pauli found:

$$R_q\psi(\mathbf{v}) := q\psi(R_{\overline{q}}(\mathbf{v})).$$

In particular, for a 360 rotation:

$$R_{-1}\psi(\mathbf{v}) = -\psi(\mathbf{v}).$$

Electrons can tell if they have been rotated 360!