Abelian Sheaves and Picard Stacks

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Outline

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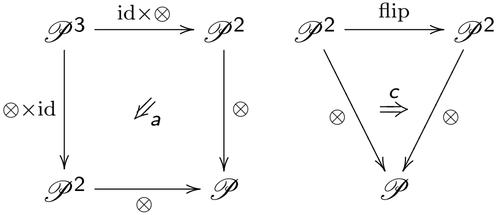
1. Motivation
2. 3-Category of Picard 2-Stacks
3. Tricategory of Complexes of Abelian Sheaves
4. Structure Theorem for Picard 2-Stacks
5. Some Ideas for Future Work

Picard Category

A Picard category is a groupoid $(\mathscr{P}, \otimes, a, c)$ where

$$\otimes : \mathscr{P} \times \mathscr{P} \longrightarrow \mathscr{P}$$

with natural transformations a and c



satisfying:

- $- \otimes X : \mathscr{P} \to \mathscr{P}$ is an equivalence for all $X \in \mathscr{P}$,
- pentagon identity,
- hexagon identities,
- $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathscr{P}$,
- $c_{X,X} = \mathrm{id}_{X \otimes X}$ for all $X \in \mathscr{P}$.

Picard Stack

A Picard stack $\mathscr{P}$ over the site S is a stack associated with a stack morphism

$$\otimes : \mathscr{P} \times \mathscr{P} \longrightarrow \mathscr{P}$$

inducing a Picard structure on $\mathscr{P}$.

Example

Let $\lambda : A^{-1} \rightarrow A^0$ be a morphism of abelian sheaves. For any $U \in \mathsf{S}$, define a groupoid $\mathscr{P}_U$ as

- *objects*: $a \in A^0(U)$
- *morphisms*: $(f, a) \in A^{-1}(U) \times A^0(U)$ such that $(f, a) : a \rightarrow a + \lambda(f)$.

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$\mathscr{P}$ is only a pre-stack $\overset{\text{stackify}}{\rightsquigarrow}$ $\mathscr{P}^{\sim}$ is a stack.

An object of $\mathscr{P}^{\sim}_U$ is a descent datum $(V_\bullet \to U, X, \varphi)$ where

- $\ldots V_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} V_1 \substack{\longrightarrow\\\longrightarrow} V_0 \xrightarrow{\delta} U$ is a hypercover,
- X is an object in $\mathscr{P}_{V_0}$,
- $\varphi : d_1^* X \to d_0^* X$ is a morphism in $\mathscr{P}_{V_1}$,

satisfying the cocycle condition in $\mathscr{P}_{V_2}$

$$d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi.$$

Example

$(V_\bullet \to U, X, \varphi)$ is effective if there exists $Y \in \mathscr{P}_U$ with isomorphism $\psi : \delta^* Y \to X$ in $\mathscr{P}_{V_0}$ compatible with φ.

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$\mathscr{P}^\sim$ is a Picard stack:

$$(V_\bullet \to U, X, \varphi) \otimes (V'_\bullet \to U, X', \varphi') = (V_\bullet \times_U V'_\bullet \to U, X + X', \varphi + \varphi')$$

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Notation: $\mathscr{P}^\sim = [A^{-1} \to A^0]^\sim$.

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Remark: Descent along hypercovers is same as descent along Čech covers - [Artin-Mazur],[Dugger-Hollander-Isaksen].

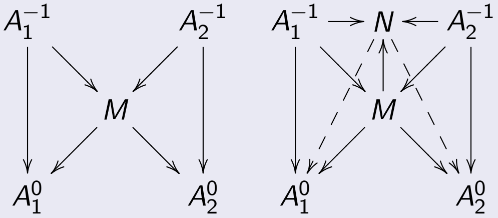
Structure Theorem for Picard Stacks

Theorem (Deligne)

The functor

$$\mathrm{ch} : \mathrm{T}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P{\scriptsize IC}}(\mathsf{S})$$

defined by sending $A^{-1} \to A^0$ *to* $[A^{-1} \to A^0]^{\sim}$ *is a biequivalence where* $\mathrm{T}^{[-1,0]}(\mathsf{S})$ *is the bicategory of complexes whose 1- and 2-morphisms are*



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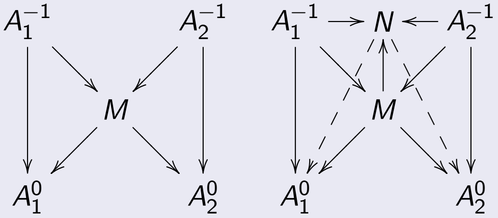


Remark: Butterflies and non-abelian version of the structure theorem by Aldrovandi and Noohi.

Derived Category

Corollary

The functor ch *induces an equivalence*

$$\mathrm{D}^{[-1,0]}(\mathsf{S}) \longrightarrow \mathrm{P{\scriptsize IC}}^{\flat}(\mathsf{S})$$

of categories where

- $\mathrm{D}^{[-1,0]}(\mathsf{S})$ *is the subcategory of the derived category of category of complexes of abelian sheaves* $A^{\bullet}$ *over a site* S *with* $H^{-i}(A^{\bullet}) \neq 0$ *only for* $i = 0, 1$
- $\mathrm{P{\scriptsize IC}}^{\flat}(\mathsf{S})$ *is the category of Picard stacks over* S *with 1-morphisms isomorphism classes of additive functors.*

Outline

2-Categorical Conventions

- A 2-category is strict and a 2-functor is a weak homomorphism.
- A 2-functor $F : \mathscr{C} \to \mathscr{D}$ is a biequivalence if it induces equivalences $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)$ for every $X, Y \in \mathscr{C}$ and every object Y' in $\mathscr{D}$ is equivalent to an object of the form $F(X)$.
- A natural 2-transformation is a strong transformation.
- A 2-groupoid is a 2-category whose 1-morphisms are invertible up to a 2-morphism and whose 2-morphisms are strictly invertible.
- A bigroupoid is a weak 2-groupoid.

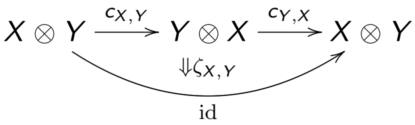
3-Categorical Conventions

- A 3-category is strict and a tricategory is a weak 3-category.
- A trihomomorphism is a weak 3-functor.
- A trihomomorphism $F : \mathscr{C} \to \mathscr{D}$ is a triequivalence if it induces biequivalences $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)$ for every $X, Y \in \mathscr{C}$ and every object Y' in $\mathscr{D}$ is biequivalent to an object of the form $F(X)$.

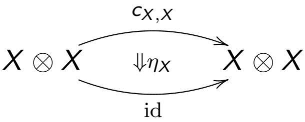
Picard 2-Category

A Picard 2-category is a 2-groupoid $(\mathbb{P}, \otimes, a, c, \pi, \mathfrak{h}_1, \mathfrak{h}_2, \zeta, \eta)$ where

- $\otimes : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$ is a 2-functor,
- a is the natural 2-transformation and π is the modification defining the monoidal structure,
- c is the natural 2-transformation and $\mathfrak{h}_1, \mathfrak{h}_2$ are the modifications defining the braiding structure,
- ζ is the modification defining the functorial 2-morphism of symmetry,



- η is the modification defining the functorial 2-morphism of Picard,



Picard 2-Category

satisfying conditions

- $- \otimes X : \mathbb{P} \to \mathbb{P}$ is a biequivalence for all $X \in \mathbb{P}$,
- $$X \otimes Y \xrightarrow{c} Y \otimes X \underset{c^2}{\overset{1}{\Downarrow \zeta}} Y \otimes X \quad = \quad X \otimes Y \underset{c^2}{\overset{1}{\Downarrow \zeta}} X \otimes Y \xrightarrow{c} Y \otimes X$$
- $\eta_X * \eta_X = \zeta_{X,X}$ for every $X \in \mathbb{P}$
- There is an additive relation between η_X,η_Y and $\eta_{X \otimes Y}$, for every $X, Y \in \mathbb{P}$,
- ...

Picard 2-Stack

A Picard 2-stack $\mathbb{P}$ is a 2-stack associated with a 2-stack morphism

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An additive 2-functor is a cartesian 2-functor compatible with the monoidal, group-like, and braiding structures.

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Picard 2-stacks over S form a 3-category, denoted by 2Pic(S), whose

- 1-morphisms are additive 2-functors,
- 2-morphisms are natural 2-transformations,
- 3-morphisms are modifications.

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Observation: natural 2-transformations are invertible up to a modifications and modifications are strictly invertible.

Example

Let $A^\bullet : A^{-2} \xrightarrow{\delta} A^{-1} \xrightarrow{\lambda} A^0$ be a complex of abelian sheaves. For any $U \in \mathsf{S}$, define a 2-groupoid $\mathbb{P}_U$ as

- *objects*: $a \in A^0(U)$,
- *1-morphisms*: $(f, a) \in A^{-1}(U) \times A^0(U)$ such that $(f, a) : a \rightarrow a + \lambda(f)$,
- *2-morphisms*: $(\sigma, f, a) \in A^{-2}(U) \times A^{-1}(U) \times A^0(U)$ such that $(\sigma, f, a) : (f, a) \Rightarrow (f + \delta(\sigma), a)$.

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$\mathbb{P}$ is a pre-pre- 2 stack $\overset{\text{stackify}\times 2}{\rightsquigarrow}$ $\mathbb{P}^\sim$ is a 2-stack.

Example

An object of $\mathbb{P}^{\sim}_{U}$ is a 2-descent datum $(V_\bullet \to U, X, \varphi, \alpha)$

- $\dots V_2 \substack{\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow} V_1 \rightrightarrows V_0 \xrightarrow{\delta} U$ is a hypercover,
- X is an object in $\mathbb{P}_{V_0}$
- $\varphi : d_0^* X \to d_1^* X$ is a 1-morphism in $\mathbb{P}_{V_1}$
- $\alpha : d_1^* \varphi \Rightarrow d_2^* \varphi \circ d_0^* \varphi$ is a 2-morphism in $\mathbb{P}_{V_2}$

satisfying the 2-cocycle condition in $\mathbb{P}_{V_3}$

$$((d_2 d_3)^* \varphi * d_0^* \alpha) \circ d_2^* \alpha = (d_3^* \alpha * (d_0 d_1)^* \varphi) \circ d_1^* \alpha$$

Example

An object of $\mathbb{P}_U^{\sim}$ is a 2-descent datum $(V_\bullet \to U, X, \varphi, \alpha)$

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$\mathbb{P}^{\sim}$ is a Picard 2-stack: $(V_\bullet \to U, X, \varphi, \alpha) \otimes (V'_\bullet \to U, X', \varphi', \alpha') =$
$(V_\bullet \times_U V'_\bullet \to U, X + X', \varphi + \varphi', \alpha + \alpha')$

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$\mathbb{P}^{\sim}$ is a Picard 2-stack: $(V_\bullet \to U, X, \varphi, \alpha) \otimes (V'_\bullet \to U, X', \varphi', \alpha') =$ $(V_\bullet \times_U V'_\bullet \to U, X + X', \varphi + \varphi', \alpha + \alpha')$

Notation: $\mathbb{P}^{\sim} = [A^\bullet]^{\sim}$

Outline

Tricategory $\mathrm{T}^{[-2,0]}(\mathsf{S})$

We are going to define...

the tricategory of length 3 complexes of abelian sheaves

$$A^\bullet : A^{-2} \longrightarrow A^{-1} \longrightarrow A^0$$

denoted by $\mathrm{T}^{[-2,0]}(\mathsf{S})$.

Hom-bigroupoid of Fractions

For any $A^\bullet$ and $B^\bullet$, the hom-bicategory of $\mathrm{T}^{[-2,0]}(\mathsf{S})$ is the bigroupoid denoted by $\mathsf{Frac}(A^\bullet, B^\bullet)$ whose

- objects,
- 1-morphisms,
- 2-morphisms,

are defined as ...

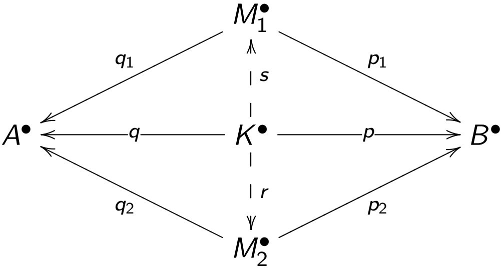
Objets of Frac($A^\bullet, B^\bullet$)

- Objects are ordered triples $(q, M^\bullet, p)$

$$A^\bullet \xleftarrow{\;q\;} M^\bullet \xrightarrow{\;p\;} B^\bullet$$

$M^\bullet$ complex, p morphism of complexes, and q quasi-isomorphism.

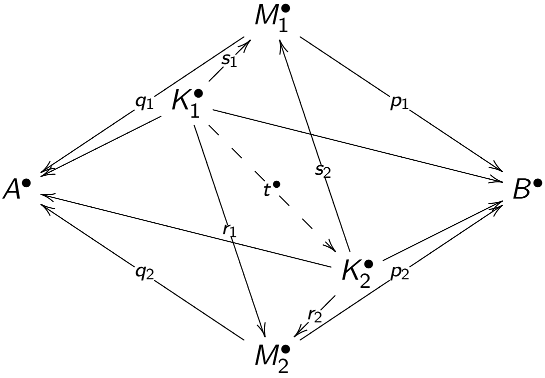
- 1-morphisms $(q_1, M_1^\bullet, p_1) \longrightarrow (q_2, M_2^\bullet, p_2)$ are ordered triples $(r, K^\bullet, s)$ such that the diagram commutes.



$K^\bullet$ complex, r and s quasi-isomorphisms.

2-morphisms of Frac($A^\bullet$, $B^\bullet$)

- 2-morphisms $(r_1, K_1^\bullet, s_1) \Rightarrow (r_2, K_2^\bullet, s_2)$ are isomorphisms $t^\bullet : K_1^\bullet \to K_2^\bullet$ of complexes such that the diagram commutes.



Strict Subcategory of $\mathrm{T}^{[-2,0]}(\mathsf{S})$

$\mathrm{T}^{[-2,0]}(\mathsf{S})$ has a strict subcategory $\mathrm{C}^{[-2,0]}(\mathsf{S})$ whose objects are same as $\mathrm{T}^{[-2,0]}(\mathsf{S})$ and whose hom-bicategory $\mathrm{Hom}_{\mathrm{C}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$ is the 2-groupoid associated to the complex

$$\mathrm{Hom}^{-2}(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{-1}(A^\bullet, B^\bullet) \longrightarrow Z^0(\mathrm{Hom}^0(A^\bullet, B^\bullet))$$

Explicitly, $\mathrm{C}^{[-2,0]}(\mathsf{S})$ has

- *objects*: complexes of abelian sheaves,
- *1-morphisms*: morphism of complexes
- *2-morphisms*: homotopies
- *3-morphisms*: homotopies of homotopies

$\mathrm{C}^{[-2,0]}(\mathsf{S})$ **is a 3-category.**

Outline

Structure Theorem for Picard 2-Stacks

Theorem (A.E.T.)

The trihomomorphism

$$2\mathrm{ch} : \mathrm{T}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P{\scriptsize IC}}(\mathsf{S})$$

defined by sending $A^\bullet$ to $[A^\bullet]^\sim$ is a triequivalence.

Proof.

- Construct the trihomomorphism $2\mathrm{ch}$ on $\mathrm{C}^{[-2,0]}(\mathsf{S})$.
- For any $A^\bullet$ and $B^\bullet$, and for any morphism $F : [A^\bullet]^\sim \to [B^\bullet]^\sim$, there exists a fraction $(q, M^\bullet, p)$ such that $F \circ 2\mathrm{ch}(q) \simeq 2\mathrm{ch}(p)$.
- Use the 2^{nd} step and the observation that $2\mathrm{ch}$ sends quasi-isomorphisms to equivalences, to extend $2\mathrm{ch}$ onto $\mathrm{T}^{[-2,0]}(\mathsf{S})$.
- Verify that $2\mathrm{ch}$ is essentially surjective.

□

Derived Category

Corollary

The functor $2\mathrm{ch}$ *induces an equivalence*

$$2\mathrm{ch}^{\flat\flat} : \mathrm{D}^{[-2,0]}(\mathsf{S}) \longrightarrow 2\mathrm{P\scriptstyle{IC}}^{\flat\flat}(\mathsf{S})$$

of categories where

- $2\mathrm{P\scriptstyle{IC}}^{\flat\flat}(\mathsf{S})$*: the* **category of Picard 2-stacks** *obtained from* $2\mathrm{P\scriptstyle{IC}}(\mathsf{S})$ *by ignoring the modifications and taking as morphisms the equivalence classes of additive 2-functors.*
- $\mathrm{D}^{[-2,0]}(\mathsf{S})$*: the* **subcategory of the derived category** *of category of complexes of abelian sheaves* $A^\bullet$ *over* S *with* $H^{-i}(A^\bullet) \neq 0$ *for* $i = 0, 1, 2$.

Proof.

$\pi_0(\mathsf{Frac}(A^\bullet, B^\bullet)) \simeq \mathrm{Hom}_{\mathrm{D}^{[-2,0]}(\mathsf{S})}(A^\bullet, B^\bullet)$ □

Remarks

- **The trihomomorphism** $2\mathrm{ch}$ **on** $\mathrm{C}^{[-2,0]}(\mathsf{S})$ **is not a triequivalence.** Not all morphisms of Picard 2-stacks are obtained from a morphism of complexes of abelian sheaves. Therefore the 1-morphisms in $\mathrm{C}^{[-2,0]}(\mathsf{S})$ are not geometric.

Remarks

- **The trihomomorphism** $2\mathrm{ch}$ **on** $\mathcal{C}^{[-2,0]}(S)$ **is not a triequivalence.** Not all morphisms of Picard 2-stacks are obtained from a morphism of complexes of abelian sheaves. Therefore the 1-morphisms in $\mathcal{C}^{[-2,0]}(S)$ are not geometric.

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	$\mathbf{T}^{[-2,0]}(\mathbf{S})$	2**Pic**(**S**)
3-categorical structure	weak	strict
objects	simple	complicated
morphisms	complicated	simple

Outline

Some Ideas for Future Work

- Tensor product of Picard 2-stacks.
- Extensions of gr-stacks by gr-stacks.
- Biextensions of length 3 complexes of abelian sheaves.