String Connections and Supersymmetric Sigma Models

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1.) What are string connections?

2.) What are string connections good for?

Setup:

- Riemannian manifold (M, g) of dimension n
- Frame bundle $P_{\mathrm{O}(n)}$

Whitehead tower:

- $\cdots \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n)$
- Orientation = Lift of $P_{\mathrm{O}(n)}$ to $\mathrm{SO}(n)$

  Obstruction: $w_1 \in \mathrm{H}^1(M, \mathbb{Z}_2)$
- Spin structure = Lift of $P_{\mathrm{SO}(n)}$ to $\mathrm{Spin}(n)$

  Obstruction: $w_2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$
- String structure = Lift of $P_{\mathrm{Spin}(n)}$ to $\mathrm{String}(n)$

  Obstruction: $\frac{1}{2}p_1 \in \mathrm{H}^4(M, \mathbb{Z})$

Question:

- A_g Levi-Cevita connection on $P_{\mathrm{O}(n)}$
- Does A_g lift along lifts

$$\cdots \longrightarrow P_{\mathrm{String}(n)} \longrightarrow P_{\mathrm{Spin}(n)} \longrightarrow P_{\mathrm{SO}(n)} \longrightarrow P_{\mathrm{O}(n)} \text{ ?}$$

Fact:

- Once a spin structure is given, A_g lifts uniquely.

What is the corresponding statement for string structures?

- **Question is not well-defined: $\mathrm{String}(n)$ is not a Lie group!**
- Way out:
  (a) use strict Fréchet Lie 2-group (Baez et al. '07)
  (b) use group object in smooth stacks (Schommer-Pries '09)
  (c) do **not** use the string group (this talk)

Fact:

- Associated to any $\mathrm{Spin}(n)$-bundle P over M is a finite-dimensional and smooth 2-gerbe $\mathbb{CS}_P$ over M, called **Chern-Simons 2-gerbe**.
- Its characteristic class is $\frac{1}{2}p_1 \in \mathrm{H}^4(M, \mathbb{Z})$.

Theorem (KW)

There is a canonical bijection

$$\left\{ \begin{array}{c} \text{Equivalence} \\ \text{classes of string} \\ \text{structures on } P \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \mathbb{CS}_P \end{array} \right\}$$

Definition (replacing previous definition)

A **string structure** on P is a trivialization

$$\mathbb{CS}_P \xrightarrow{\mathbb{T}} \mathbb{I}$$

of the Chern-Simons 2-gerbe. Here $\mathbb{I}$ is the trivial gerbe.

Another fact:

- Associated to a connection A on P is a connection ∇_A on the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

Definition

A **string connection** for $(\mathbb{T}, A)$ is an extension of the string structure $\mathbb{T}$ to a **connection-preserving** trivialization

$$(\mathbb{CS}_P, \nabla_A) \xrightarrow{(\mathbb{T},\blacktriangledown)} (\mathbb{I}, \nabla_H).$$

The connection ∇_H on $\mathbb{I}$ is given by $H \in \Omega^3(M)$.

Theorem (KW)

For any pair $(\mathbb{T}, A)$ there exists a string connection. Their choices form an affine space.

What are string connections?

- Connection-preserving trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

What are they good for?

Transgression:

- Σ a closed surface
- **Transgression** is a functorial assignment

$$\left\{ \begin{array}{c} \text{2-gerbes over} \\ M \text{ with} \\ \text{connection} \end{array} \right\} \xrightarrow{\mathscr{T}} \left\{ \begin{array}{c} \text{hermitian line bundles} \\ \text{over } C^\infty(\Sigma, M) \text{ with} \\ \text{connection} \end{array} \right\}$$

- On characteristic classes, it is **integration along the fibre**:

$$\mathrm{H}^4(M, \mathbb{Z}) \xrightarrow{\int_\Sigma \mathrm{ev}^*} \mathrm{H}^2(C^\infty(\Sigma, M), \mathbb{Z})$$

What is the transgression of the Chern-Simons 2-gerbe?

- It is a **Pfaffian line bundle** over $C^\infty(\Sigma, M)$.

What does the string connection $(\mathbb{T}, \blacktriangledown)$ do?

- By functorality, it **trivializes** this Pfaffian line bundle.

Pfaffian bundle:

- (Σ, γ) Riemann surface with spin structure, $S\Sigma$ spinor bundle.
- For each map $X : \Sigma \longrightarrow M$, there is a **twisted Dirac operator**

$$D_X : \Gamma(S\Sigma \otimes X^* TM) \longrightarrow \Gamma(S\Sigma \otimes X^* TM).$$

- The **Pfaffian** of D_X is a complex line.

Theorem (Freed '87)

These Pfaffians form a hermitian line bundle $\mathcal{P}\mathit{faff}(D)$ with connection over $C^\infty(\Sigma, M)$.

Theorem (Bunke '09)

There exists a canonical isomorphism

$$\mathscr{T}(\mathbb{CS}_P, \nabla_A) \cong \mathcal{P}\mathit{faff}(D).$$

A **supersymmetric sigma model** is given by:

- Riemannian manifold (M, g)
- string structure $\mathbb{T}$ with string connection $\blacktriangledown$ for $(\mathbb{T}, A_g)$.

A field is:

- Riemann surface (Σ, γ) with spin structure
- map $X : \Sigma \longrightarrow M$ **"Boson"**
- section $\psi \in \Gamma(S\Sigma \otimes X^*TM)$ **"Fermion"**

Action functional:

- $S_\Sigma(X, \psi) := \int_\Sigma \mathrm{dvol}_\gamma \left\{ \frac{1}{2} \langle \mathrm{d}X, \mathrm{d}X \rangle_g + \langle \psi, D_X \psi \rangle_g \right\}$

What does the string connection do?

- The **fermionic path integral** is a well-defined element

$$s(X) := \int \mathrm{d}\psi \;\; \mathrm{e}^{\int_\Sigma \, \mathrm{dvol}_\gamma \, \langle \psi, D_X \psi \rangle_g} \in \mathcal{P}\mathit{faff}(D)$$

  (Freed-Moore '06)
- The remaining **Feynman amplitude**

$$\mathcal{A}_\Sigma(X) := \mathrm{e}^{\int_\Sigma \, \mathrm{dvol}_\gamma \, \frac{1}{2} \langle \mathrm{d}X, \mathrm{d}X \rangle_g} \cdot s(X)$$

  is a section $\mathcal{A}_\Sigma \in \Gamma(\mathcal{P}\mathit{faff}(D))$.
- Since the string connection trivializes $\mathcal{P}\mathit{faff}(D)$, the **section** $\mathcal{A}_\Sigma$ becomes a **function**

$$\mathcal{A}_\Sigma : C^\infty(\Sigma, M) \longrightarrow \mathbb{C}.$$

What are string connections?

- Connection-preserving trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

What is a string connection good for?

- It trivializes the Pfaffian line bundle $\mathcal{P}\mathit{faff}(D)$, and makes the Feynman amplitude of the supersymmetric sigma model a function.

  In terminology of Freed and Moore, it "sets the quantum integrand".

Literature:

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