We divide the remaining cases into four subcases according to whether $\alpha + \alpha_i$ and $\beta + \alpha_i$ are elements of Φ or not, and show that $e_i v_{\alpha,\beta} = 0$. We first consider the case where one of $\alpha+\alpha_i$, $\beta+\alpha_i$ belongs to Φ. As the argument is the same, we only treat the case where $\alpha + \alpha_i \in \Phi$. When $\alpha + \alpha_i \in \Phi$, we can write $[E_{\alpha_i}, E_\alpha] = c_{\alpha_i,\alpha} E_{\alpha_i+\alpha}$ with a non-zero number $c_{\alpha_i,\alpha}$. Thus,

$$\begin{aligned}[L_{\alpha_i}, L_\alpha] &= e_i \iota(E_\alpha) = \iota(e_i E_\alpha) = \iota([E_{\alpha_i}, E_\alpha]) \\ &= c_{\alpha_i,\alpha}\iota(E_{\alpha_i+\alpha}) = c_{\alpha_i,\alpha} L_{\alpha_i+\alpha}.\end{aligned}$$

This means that the element

$$[[L_{\alpha_i}, L_\alpha], L_\beta] - \iota([[E_{\alpha_i}, E_\alpha], E_\beta])$$

is equal to a scalar multiple of $v_{\alpha+\alpha_i,\beta}$. By the maximality of γ, this element is 0. Using this we have

$$e_i v_{\alpha,\beta} = [L_\alpha, [L_{\alpha_i}, L_\beta]] - \iota([E_\alpha, [E_{\alpha_i}, E_\beta]]).$$

If $\beta + \alpha_i \notin \Phi$, then $\beta + \alpha_i \neq 0$ implies that $[E_{\alpha_i}, E_\beta] = 0$. So

$$[L_{\alpha_i}, L_\beta] = e_i \iota(E_\beta) = \iota(e_i E_\beta) = \iota([E_{\alpha_i}, E_\beta]) = 0,$$

which implies that $e_i v_{\alpha,\beta} = 0$. If $\beta+\alpha_i \in \Phi$, then $e_i v_{\alpha,\beta}$ is equal to a scalar multiple of $v_{\alpha,\beta+\alpha_i}$ and the maximality of γ again implies that $e_i v_{\alpha,\beta} = 0$.

If both $\alpha + \alpha_i, \beta + \alpha_i$ are not in Φ, then

$$\begin{aligned}[L_{\alpha_i}, L_\alpha] &= e_i\iota(E_\alpha) = \iota(e_i E_\alpha) = \iota([E_{\alpha_i}, E_\alpha]) = 0, \\ [L_{\alpha_i}, L_\beta] &= e_i\iota(E_\beta) = \iota(e_i E_\beta) = \iota([E_{\alpha_i}, E_\beta]) = 0.\end{aligned}$$

Hence we have $e_i v_{\alpha,\beta} = 0$ in this case also.

Since i is arbitrary and $e_i v_{\alpha,\beta} = 0$, Assertion 3(3) tells us that $v_{\alpha,\beta}$ is equal to a scalar multiple of $\iota(E_{1n})$. In particular, we have

$$\gamma = \alpha_1 + \cdots + \alpha_{n-1}$$

and $\{\alpha, \beta\} = \{\alpha_1 + \cdots + \alpha_k, \alpha_{k+1} + \cdots + \alpha_{n-1}\}$ for some k. However, $[L_\alpha, L_\beta]$ is equal to L_{1n} in this case and so $v_{\alpha,\beta} = 0$, which contradicts our choice of γ. Hence, we have $\mathcal{A} = \emptyset$ and Assertion 4 follows.

We are now ready to prove Theorem 2.2. We have constructed a map $\iota : \mathfrak{g} \to U$ which satisfies $[\iota(X), \iota(Y)] = \iota([X, Y])$. Our next task is to prove the universality of the pair (U, ι); however, this is obvious because the map $\phi : U \to A$ is uniquely determined by the requirements that $\phi(e_i) = \rho(E_{i,i+1})$ etc. $\square$

2.2. The quantum algebra of type A_{r-1}

Based on Theorem 2.2 Drinfeld and Jimbo introduced the quantum algebra which is obtained as a "deformation"of the enveloping algebra of $\mathfrak{sl}_r = \mathfrak{sl}(r, \mathbb{C})$. The definition is as follows. We choose $\mathbb{Q}(v)$ as a base field since it is not necessary to assume it to be $\mathbb{C}(v)$. The element t_i is often denoted by v^{h_i} and $\alpha_j(h_i) = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$ by definition.

DEFINITION 2.5. Let $K = \mathbb{Q}(v)$ where v is an indeterminate. The **quantum algebra of type A_{r-1}** is the unital associative K-algebra $U_v(\mathfrak{sl}_r)$ defined by the following generators and relations.

Generators: $t_i^{\pm 1}, e_i, f_i \quad (1 \le i \le r-1)$.

Relations:

$$t_i e_j t_i^{-1} = v^{\alpha_j(h_i)} e_j, \quad t_i f_j t_i^{-1} = v^{-\alpha_j(h_i)} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{v - v^{-1}},$$

$$[t_i, t_j] = 0, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1,$$

$$e_i^2 e_j - (v + v^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \ (i - j = \pm 1),$$
$$e_i e_j = e_j e_i \ (otherwise),$$

$$f_i^2 f_j - (v + v^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \ (i - j = \pm 1),$$
$$f_i f_j = f_j f_i \ (otherwise).$$

These relations are called the (deformed) Serre relations.

DEFINITION 2.6. Let $[k] = \frac{v^k - v^{-k}}{v - v^{-1}}$, for $k \in \mathbb{N}$, and let $[n]! = \prod_{k=1}^{n} [k]$. Then $f_i^{(n)}$ is defined by

$$f_i^{(n)} = \frac{f_i^n}{[n]!}.$$

Roughly speaking, the quantum algebra is the algebra which is obtained by "integrating" the Cartan subalgebra and deforming the other relations "nicely".

We may obtain representations of $U_v(\mathfrak{sl}_r)$ by deforming the representations of $\mathfrak{g}$. We can also define tensor product representations by deforming the coproduct of the enveloping algebra as follows.

$$\Delta(t_i) = t_i \otimes t_i, \ \Delta(e_i) = 1 \otimes e_i + e_i \otimes t_i^{-1},$$
$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i.$$

EXERCISE 2.7. Verify that Δ defines an algebra homomorphism from $U_v(\mathfrak{sl}_r)$ to $U_v(\mathfrak{sl}_r) \otimes U_v(\mathfrak{sl}_r)$.

EXERCISE 2.8. Let $V = K^r$ and define $\rho : U_v(\mathfrak{sl}_r) \to \mathrm{End}(V)$ by

$$\rho(t_i) = \mathrm{I} + (v - 1)E_{i,i} + (v^{-1} - 1)E_{i+1,i+1},$$
$$\rho(e_i) = E_{i,i+1}, \ \rho(f_i) = E_{i+1,i}.$$

Show that (ρ, V) is a representation of $U_v(\mathfrak{sl}_r)$. This is called the natural (or vector) representation of $U_v(\mathfrak{sl}_r)$.

EXERCISE 2.9. Let V be the natural representation of $U_v(\mathfrak{sl}_2)$. Decompose $V \otimes V$ into a sum of irreducible $U_v(\mathfrak{sl}_2)$-submodules.