to ensure that the categories of quantum spaces and quantum groups have reasonable properties, it would be necessary to impose some restrictions on the class of algebras which are acceptable as 'quantized algebras of functions'. Manin suggests that one should work with 'Koszul algebras', but we shall not discuss this point here.) As is common practice in the literature, we shall often abuse terminology by referring to a Hopf algebra itself as a quantum group.

As the preceding discussion suggests, one way to try to construct non-classical examples of quantum groups is to look for deformations, in the category of Hopf algebras, of classical algebras of functions $\mathcal{F}(G)$. Just as the classical Poisson bracket can be recovered as the 'first order part' of Moyal's deformation (see (3)), so it turns out that the existence of a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ automatically endows the group G itself with extra structure, namely that of a *Poisson–Lie group*. This is a Poisson structure on G which is compatible with the group structure in a certain sense. Conversely, to construct deformations of $\mathcal{F}(G)$, it is natural to begin by describing the possible Poisson–Lie group structures on G and then to attempt to extend these 'first order deformations' to full deformations. This is the approach taken in this book. Poisson–Lie groups are also of interest in their own right, for they form the natural setting for the study of classical integrable systems with symmetry.

There is another Hopf algebra associated to any Lie group G, namely the universal enveloping algebra $U(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. This is essentially the *dual* of $\mathcal{F}(G)$ in the category of Hopf algebras. In general, the vector space dual A^* of any finite-dimensional Hopf algebra A is also a Hopf algebra: the multiplication $A^* \otimes A^* \to A^*$ is dual to the comultiplication $\Delta : A \to A \otimes A$ of A, and the comultiplication of A^* is dual to the multiplication of A. Note that A^* is commutative if and only if A is cocommutative, i.e. if and only if $\Delta(A)$ is contained in the symmetric part of $A \otimes A$. If, as is usually the case in examples of interest, A is infinite dimensional, this duality often continues to hold provided the dual and tensor product are defined appropriately. To a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ through (not necessarily commutative) Hopf algebras therefore corresponds a deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ through (not necessarily cocommutative) Hopf algebras.

In fact, only non-cocommutative deformations of $U(\mathfrak{g})$ are of interest, since any deformation of $U(\mathfrak{g})$ through cocommutative Hopf algebras is necessarily of the form $U(\mathfrak{g}_h)$ for some deformation $\mathfrak{g}_h$ of $\mathfrak{g}$ through Lie algebras. However, many interesting Lie algebras have no non-trivial deformations. This is the case, for example, if $\mathfrak{g}$ is a (finite-dimensional) complex semisimple Lie algebra, such as the Lie algebra $sl_2(\mathbb{C})$ of 2×2 complex matrices of trace zero. This follows from the fact that the condition of semisimplicity is open, so that any small deformation of $\mathfrak{g}$ will still be semisimple, whereas the semisimple Lie algebras are discretely parametrized (by their Dynkin diagrams, for example).

The first example of a non-cocommutative deformation of this type was discovered by P. P. Kulish and E. K. Sklyanin in 1981 in the case $\mathfrak{g} = sl_2(\mathbb{C})$ (although the importance of its Hopf structure was not realized until later). Note that $sl_2(\mathbb{C})$ has a basis

$$\bar{X}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{X}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

whose Lie brackets are given by

$$[\bar{X}^+, \bar{X}^-] = \bar{H}, \quad [\bar{H}, \bar{X}^\pm] = \pm 2\bar{X}^\pm. \tag{5a}$$

The comultiplication is given on these basis elements by

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H}, \quad \Delta(\bar{X}^\pm) = \bar{X}^\pm \otimes 1 + 1 \otimes \bar{X}^\pm, \tag{5b}$$

an assignment which extends uniquely to an algebra homomorphism $\Delta : U(sl_2(\mathbb{C})) \to U(sl_2(\mathbb{C})) \otimes U(sl_2(\mathbb{C}))$. The deformation $U_h(sl_2(\mathbb{C}))$ is generated by elements H, $X^\pm$, which satisfy the relations

$$X^+X^- - X^-X^+ = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \quad HX^\pm - X^\pm H = \pm 2X^\pm. \tag{6a}$$

It has a non-cocommutative comultiplication given on generators by

$$\begin{gathered} \Delta(H) = H \otimes 1 + 1 \otimes H, \\ \Delta(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+, \quad \Delta(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-. \end{gathered} \tag{6b}$$

Formally, at least, it is clear that (6a) and (6b) go over into (5a) and (5b) as $h \to 0$. The Hopf algebra defined in (6a,b) is called 'quantum $sl_2(\mathbb{C})$'. (See Chapter 6 for the formulas for the antipode and counit of $U_h(sl_2(\mathbb{C}))$, and for a way to make sense of expressions such as e^{hH}.)

The Hopf algebra dual to $U_h(sl_2(\mathbb{C}))$, the 'algebra $\mathcal{F}_h(SL_2(\mathbb{C}))$ of functions on quantum $SL_2(\mathbb{C})$', was discovered by L. D. Faddeev and L. A. Takhtajan in 1985. It is the associative algebra generated by elements a, b, c, d with the following multiplicative relations:

$$ab = e^{-h}ba, \quad ac = e^{-h}ca, \quad bd = e^{-h}db, \quad cd = e^{-h}dc, \tag{7}$$

$$bc = cb, \quad ad - da + (e^h - e^{-h})bc = 0, \tag{8}$$

$$ad - e^{-h}bc = 1, \tag{9}$$

and comultiplication

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.$$

Note that, when $h \to 0$, the relations (7), (8) and (9) just say that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has commuting entries and determinant one. Thus, $\mathcal{F}_h(SL_2(\mathbb{C}))$ is a deformation of the algebra of functions on the group $SL_2(\mathbb{C})$ of 2×2 complex matrices of determinant one.

As we mentioned at the beginning of this introduction, the algebra structure of $\mathcal{F}_h(G)$ can be described by a matrix of constants, namely

$$R = e^{-h/2} \begin{pmatrix} e^h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & e^h - e^{-h} & 1 & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}. \tag{10}$$

In fact, if

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the relations (7) and (8) are equivalent to

$$(T \otimes 1)(1 \otimes T)R = R(1 \otimes T)(T \otimes 1). \tag{11}$$

Note that $T \otimes 1$ and $1 \otimes T$ do not commute, since the entries of T do not commute (if $h \neq 0$); note also that R is most naturally viewed as an element of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. It is in the form (11) that quantum groups usually appear in the theory of integrable systems.

For the dual Hopf algebra $U_h(sl_2(\mathbb{C}))$, the quantum R-matrix expresses, as one would expect, the non-cocommutativity of the comultiplication. Namely, let $\Delta^{\mathrm{op}}(x)$ be the result of interchanging the order of the factors in $\Delta(x)$, for any $x \in U_h(sl_2(\mathbb{C}))$. It turns out that there is an invertible element $\mathcal{R} \in U_h(sl_2(\mathbb{C})) \otimes U_h(sl_2(\mathbb{C}))$, called the 'universal R-matrix', such that

$$\Delta^{\mathrm{op}}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}$$

for all $x \in U_h(sl_2(\mathbb{C}))$ (actually, $\mathcal{R}$ is a formal infinite sum of elements of the algebraic tensor product). The relation between $\mathcal{R}$ and R is very simple: the reader will easily verify that, if we replace $\tilde{X}^{\pm}$ and $\tilde{H}$ by $X^{\pm}$ and H in (4), we obtain a matrix representation of $U_h(sl_2(\mathbb{C}))$; applying this representation to $\mathcal{R}$ gives the matrix R.

Quantum groups might have remained a curiosity to the mathematical community at large but for their surprising connections with other parts of mathematics, most notably the theory of invariants of links and 3-manifolds, and the representation theory of Lie algebras in characteristic p.

The former depends on the classical relation between braids and links. Recall that a *braid on m strands* is a collection of m non-intersecting strings in $\mathbb{R}^3$ joining m fixed points in a plane to m fixed points in another parallel plane. Joining corresponding points in the two planes in a standard way associates to any braid a link (called its 'closure'), i.e. a collection of non-intersecting circles in $\mathbb{R}^3$. Joining braids end to end makes the set of isotopy classes of braids into a group $\mathcal{B}_m$. The relation with quantum groups arises because there is a simple way to associate to any quantum R-matrix $R \in \mathrm{End}(V \otimes V)$ a representation ρ_m of $\mathcal{B}_m$ on $V^{\otimes m}$ for all $m \geq 2$. This depends on the fact that R satisfies the *quantum Yang–Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12};$$

here, R_{12} means $R \otimes \mathrm{id} \in \mathrm{End}(V^{\otimes 3})$, etc. To obtain an invariant of links, one needs a family of 'traces' $tr_m : \mathrm{End}(V^{\otimes m}) \to \mathbb{C}$ such that $tr_m(\rho_m(b)) = tr_n(\rho_n(b'))$ whenever the closures of the braids $b \in \mathcal{B}_m$ and $b' \in \mathcal{B}_n$ are equivalent links. Thanks to a classical theorem of A. Markov, it is known precisely which pairs (b, b') have the latter property (and for this reason, the tr_m are usually called 'Markov traces'). Using the quantum R-matrix (10) and a suitable Markov trace, one obtains in this way the celebrated *Jones polynomial*. In fact, this is essentially Jones's original construction, except that he obtained his R-matrix by using a 'Hecke algebra' instead of a quantum group (but we shall see that Hecke algebras should probably be regarded as 'quantum' objects).

The application to 3-manifolds is based on the well-known fact that every compact, oriented, connected 3-manifold without boundary can be obtained, up to homeomorphism, by performing surgery on a link in the 3-dimensional sphere. One shows that a cleverly chosen combination of the quantum invariants of this link depends only on the 3-manifold, and not on the choice of the link along which surgery is performed.

The application of quantum groups to representations of Lie algebras in characteristic p is no less remarkable. It makes use of a certain 'standard' deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$, where $\mathfrak{g}$ is any finite-dimensional complex semisimple Lie algebra (and which reduces, when $\mathfrak{g} = sl_2(\mathbb{C})$, to the algebra found by Kulish and Sklyanin). To describe the relation with characteristic p, it is convenient to replace the deformation parameter h by $\epsilon = e^h$, and to write $U_\epsilon(\mathfrak{g})$ for $U_h(\mathfrak{g})$. It then turns out that the representation theory of $U_\epsilon(\mathfrak{g})$ depends crucially on whether ϵ is a root of unity or not. In the latter case, the theory is essentially the same as the representation theory of $\mathfrak{g}$ itself (over $\mathbb{C}$), but in the former it resembles the modular representation theory of $\mathfrak{g}$. This is more than an analogy: if ϵ is a primitive pth root of unity, where p is a prime, there is a ring homomorphism from $U_\epsilon(\mathfrak{g})$ to the enveloping algebra $U_{\mathbb{F}_p}(\mathfrak{g})$