5) $q_{ij} = 1$ *for all distinct* $i, j \in I := \{l \neq a: q_{al} = 1\}$: Indeed, $z' = z - e^a + e^i + e^j + e^n$ is >0 and satisfies $z'_n = s + 1$, hence $2 \leqslant q(z') = 5 - q(z|e^a) - q_{al} - q_{aj} - q_{an} + q_{ij} + q_{in} + q_{jn} = 1 + q_{ij}$.

6) $z_n = 3 + \sum_{i \in I} z_i = -1 + \sum_{j \in J} z_j$, *where* $J := \{j: q_{aj} = 0\}$: For, $1 = q(z|e^a) = 2z_a + \sum_{i \in I} z_i - z_n$ and $0 = q(z|e^n) = -z_a - \sum_{i \in I} z_i - \sum_{j \in J} z_j + 2z_n = (z_n - \sum_{i \in I} z_i) - z_a - \sum_{j \in J} z_j + z_n = 3 - 2 - \sum_{j \in J} z_j + z_n$.

7) $z_i = 1$ *and* $q_{ij} = 0$ *if* $i \in I$ *and* $j \in J$: For, $0 = q(z|e^i) = z_a + 2z_i + \sum_{i \neq l \in I} z_l + \sum_{j \in J} z_j q_{ij} - z_n = (\sum_{l \in I} z_l - z_n) + z_a + z_i + \sum_{j \in J} z_j q_{ij} = -1 + z_i + \sum_{j \in J} z_j q_{ij}$.

8) *If* $c \in J$ *is such that* $z_c \geqslant z_j$ *for all* $j \in J$, *there are two distinct* j, l *in* J *such that* $q_{cj} = q_{cl} = 0$: For, $0 = q(z|e^c) = 2z_c + \sum_{c \neq j \in J} z_j q_{cj} - z_n = z_c - \sum_{j \in J'} z_j + (\sum_{j \in J} z_j - z_n) = z_c + 1 - \sum_{j \in J'} z_j$, where $J' = \{j \in J: q_{cj} = 0\}$. Since $z_c \geqslant z_j$ for each $j \in J'$, J' has at least 2 elements.

9) *Conclusion*: Suppose that $s = z_n \geqslant 7$. By 6) and 7), I has at least 4 elements i_1, i_2, i_3, i_4. The "full" subbigraph of the bigraph of q which is formed by the vertices $a, i_1, i_2, i_3, i_4, c, j, l, n$ is therefore isomorphic to the fourth bigraph of Fig. 1 or to a bigraph having one broken edge less (between j and l). This leads us to the contradiction

$$q(e^a + e^{i_1} + e^{i_2} + e^{i_3} + e^{i_4} + 3e^c + 2e^j + 2e^l + 6e^n) = -4 + 4q_{jl} \leqslant 0. \surd$$

6.8. Remarks and References

1. In [135, 1977], A.V. Roiter examines functions on $\mathbb{Z}^n$ of the form $q(x) = \sum_i q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$, where q_i, q_{ij}/q_i and q_{ij}/q_j are integers and $q_i > 0$.
2. The roots of a positive unit form provide a system of roots in the sense of [29, Bourbaki, 1968]. Dynkin graphs were introduced in [44, Dynkin, 1947].
3. [118, Ovsienko, 1978].
4. [75, von Höhne, 1988].
5. See [135, Roiter, 1977], where a more precise theorem is proved for integral quadratic forms.
6. [119, Ovsienko, 1979]. Ovsienko proves his theorem in the more general context of Roiter's integral forms. His key idea is to reduce the proof to the case of an L-bigraph (which satisfies the statements 1), 2) and 3) of 6.7). The faithful weakly positive L-bigraphs (= sincere weakly positive graphical forms) are classified in [132, Ringel, 1984]. We owe the details of our proof to K. Bongartz who leans on Ringel.

7. Representations of Quivers

In this section, we examine representations of a *finite* quiver Q over the algebraically closed field k. These representations are identified with left modules over the k-category of paths kQ. Unless otherwise stated, we assume that they are pointwise finite.

7.1. In our investigation, a central rôle is played by the quadratic form[1] $q_Q: \mathbb{Z}^{Q_v} \to \mathbb{Z}$ defined by

$$q_Q(d) = \sum_{a \in Q_v} d(a)^2 - \sum_{\alpha \in Q_a} d(t\alpha)d(h\alpha).$$

We are especially interested in the value $q_Q(\underline{\dim}\, V)$ of q_Q at the *dimension-function* $\underline{\dim}\, V: x \mapsto \dim V(x)$ of a representation V.

Theorem[2]. *The number of isoclasses of indecomposable representations of* Q *is finite if and only if* q_Q *is positive definite. If this is the case, the map* $V \mapsto \underline{\dim}\, V$ *provides a bijection between the set of these isoclasses and the set of positive roots of* q_Q.

As we know by 6.2, q_Q is positive definite if and only if Q is a disjoint union of Dynkin quivers. It follows that, if Q is Dynkin of type A_n, it has $\frac{1}{2}n(n+1)$ isoclasses of indecomposables. Among them, one only is *omnipresent*[3]. It may be delineated as follows:

$$k \overset{1}{—} k \overset{1}{—} k — \cdots — k \overset{1}{—} k$$

If Q is Dynkin of type D_n, it has $(n-1)n$ isoclasses of indecomposables. Up to isomorphism, the omnipresent indecomposables are those of Fig. 1 (according to the orientations of the arrows, a is represented by the matrix $[0\ \ 1]$ or $[1\ \ 0]^T$, b by $[1\ \ 0]$ or $[0\ \ 1]^T$ and c by $[1\ \ 1]$ or $[1\ \ 1]^T$):

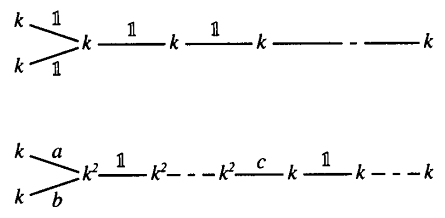

Fig. 1

If Q is Dynkin of type E_6, E_7 or E_8, it has 36, 63 or 120 isoclasses of indecomposables respectively. A concrete description will be produced in Sect. 10.

Example 1. A vector space V_0 together with 3 subspaces V_1, V_2, V_3 can be interpreted as a representation of the quiver $\begin{smallmatrix} x_1 \searrow \\ x_2 \nearrow \end{smallmatrix} x_0 \leftarrow x_3$. These representations can easily be classified "by hand". In particular, when V_0 has dimension 4 and V_1, V_2, V_3 are pairwise supplementary subspaces of dimension 2, the associated representation is a direct sum of 2 isomorphic indecomposables with dimension-function $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\!\!>\!2—1$. In geometrical terms, this means that in the projective 3-space

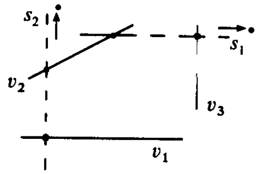

Fig. 2

three straight lines v_1, v_2, v_3 in skew position admit two common secants s_1, s_2 in skew position (Fig. 2).

Example 2. Let $V_n = k^n$ be the space of n-columns and V_i the i-dimensional subspace $\{x \in k^n\colon x_q = 0 \text{ if } i < q\}$. Each invertible $n \times n$-matrix g gives rise to the representation

$$V_1 \to V_2 - \cdots \to V_{n-1} \to V_n \leftarrow gV_{n-1} \leftarrow \cdots - gV_2 \leftarrow gV_1$$

of a Dynkin quiver of type A_{2n-1} (the maps are inclusions). Our classification of the indecomposables here means that V_n admits a basis $b^1, \ldots, b^n$ such that $b^i \in V_i \cap gV_{\sigma i}$, $V_i = \bigoplus_{h \leqslant i} kb^h$ and $gV_j = \bigoplus_{\sigma h \leqslant j} kb^h$ for some permutation σ (in terms of algebraic groups, two Borel subgroups of GL_n contain a common maximal torus[4]). Denoting by $e^1, \ldots, e^n$ the natural basis of k^n, by $\underline{\sigma}$ the permutation-matrix such that $\underline{\sigma}e^i = e^{\sigma i}$ and by $\underline{b}$ the upper triangular matrix with columns $b^1, \ldots, b^n$, we get $\underline{b}e^i = b^i$, $\underline{b}^{-1}gV_j = \bigoplus_{\sigma h \leqslant j} k\underline{b}^{-1}b^h = \bigoplus_{\sigma h \leqslant j} ke^h = \underline{\sigma}^{-1}V_j$ and $\underline{\sigma}\underline{b}^{-1}gV_i \subset V_i$. We infer that $g = \underline{b}\underline{\sigma}^{-1}\underline{c}$, where $\underline{\sigma}^{-1}$ is a permutation-matrix and $\underline{b}, \underline{c}$ are upper triangular.

7.2. Among the possible *proofs* of Theorem 7.1, we choose one which stresses the rôle of the quadratic form q_Q. It uses elementary notions of algebraic geometry and homological algebra.

We first notice that two representations V and W of Q give rise to an exact sequence

$$(*)\quad 0 \to \mathrm{Hom}(V, W) \xrightarrow{\gamma} \prod_{x \in Q_v} \mathrm{Hom}_k(V(x), W(x)) \xrightarrow{\delta} \prod_{\alpha \in Q_a} \mathrm{Hom}_k(V(t\alpha), W(h\alpha)) \to$$
$$\xrightarrow{\varepsilon} \mathrm{Ext}^1(V, W) \to 0,$$

where γ denotes the inclusion, δ maps a family $(f(x))_{x \in Q_v}$ onto $(f(h\alpha)V(\alpha) - W(\alpha)f(t\alpha))_{\alpha \in Q_a}$ and ε maps $(g(\alpha))_{\alpha \in Q_a}$ onto the equivalence class of the exact sequence $0 \to W \xrightarrow{i} E \xrightarrow{p} V \to 0$ such that $E(x) = W(x) \oplus V(x)$ for each $x \in Q_v$ and $E(\alpha) = \begin{bmatrix} W(\alpha) & g(\alpha) \\ 0 & V(\alpha) \end{bmatrix}$ for each $\alpha \in Q_a$ (the morphisms i and p are the obvious ones).

If d and e are the dimension-functions of V and W, $(*)$ implies

$$\begin{aligned}\dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W) &= \sum_x \dim \mathrm{Hom}_k(V(x), W(x)) \\ &\quad - \sum_\alpha \dim \mathrm{Hom}_k(V(t\alpha), W(h\alpha)) \\ &= \sum_x d(x)e(x) - \sum_\alpha d(t\alpha)e(h\alpha)\end{aligned}$$

and in particular

$$\dim \mathrm{Hom}(V, V) - \dim \mathrm{Ext}^1(V, V) = \sum_x d(x)^2 - \sum_\alpha d(t\alpha)d(h\alpha) = q_Q(d).$$

Lemma. *If q_Q is positive definite, we have* $\mathrm{Hom}(V, V) = k\mathbb{1}_V$ *for each indecomposable representation V of Q.*

Proof[5]. Let V be a counterexample of minimal dimension and f a non-zero nilpotent endomorphism of V whose image I has minimal dimension. Set $K = \mathrm{Ker} f = K_1 \oplus \cdots \oplus K_s$, where each K_i is indecomposable.

Since $\dim I$ is minimal, I is indecomposable, we have $f^2 = 0$, hence $I \subset K$, and each non-zero projection $p_i\colon I \to K_i$ is injective. Since V is indecomposable, the equivalence class $\varepsilon = (\varepsilon_i) \in \mathrm{Ext}^1(I, K) \xrightarrow{\sim} \bigoplus_i \mathrm{Ext}^1(I, K_i)$ of the exact sequence $0 \to K \to V \to I \to 0$ is non-zero, and so is each ε_i.

Now, since $p_i\colon I \to K_i$ is injective, the exact sequences (*) applied to $V = K_i$, I and $W = K_i$ show that $\mathrm{Ext}^1(p_i, K_i)\colon \mathrm{Ext}^1(K_i, K_i) \to \mathrm{Ext}^1(I, K_i)$ is surjective. It follows that $\mathrm{Ext}^1(K_i, K_i) \neq 0$. On the other hand, the minimality of $\dim V$ implies $\mathrm{Hom}(K_i, K_i) = k\mathbb{1}_{K_i}$, hence the required contradiction

$$\begin{aligned}0 < q_Q(\underline{\dim}\, K_i) &= \dim \mathrm{Hom}(K_i, K_i) - \dim \mathrm{Ext}^1(K_i, K_i) \\ &= 1 - \dim \mathrm{Ext}^1(K_i, K_i) \leqslant 0. \checkmark\end{aligned}$$

7.3. Proof of theorem 7.1. With the notations of 7.2, suppose that $V = W$ and that $V(x) = k^{d(x)}$ for each $x \in Q_v$. The representation V can then be identified with the family

$$(V(\alpha))_{\alpha \in Q_a} \in \prod_\alpha \mathrm{Hom}_k(V(t\alpha), V(h\alpha)) \xrightarrow{\sim} \prod_\alpha k^{d(h\alpha) \times d(t\alpha)}.$$

We denote this product by X_d and endow it with its natural structure of an algebraic variety of dimension $\sum_\alpha d(t\alpha)d(h\alpha)$.

On the other hand, the space $\prod_x \mathrm{Hom}_k(V(x), V(x))$ of $(*)$ is identified with a product of matrix-algebras $\prod_x k^{d(x) \times d(x)}$. Its invertible elements form an algebraic group $G_d = \prod_x GL_{d(x)}$ of dimension $\sum_x d(x)^2$. The formula $(gV)(\alpha) = g(h\alpha)V(\alpha)g(t\alpha)^{-1}$ defines an action of G_d on X_d whose orbits correspond bijectively to the isoclasses of representations of Q with dimension-function d.

The *isotropy group* $G_{dV} = \{g \in G_d\colon gV = V\}$ is the group of automorphisms of V, i.e. of invertible elements of $\mathrm{Hom}(V, V)$. It is Zariski-open in $\mathrm{Hom}(V, V)$ and has the same dimension. It follows[6] that the orbit $G_dV = \{gV\colon g \in G_d\}$ has the dimension $\dim G_dV = \dim G_d - \dim G_{dV} = \dim G_d - \dim \mathrm{Hom}(V, V)$ and that

$$\dim \mathrm{Hom}(V, V) - \dim \mathrm{Ext}^1(V, V) = q_Q(d) = \dim G_d - \dim X_d$$
$$= \dim \mathrm{Hom}(V, V) - (\dim X_d - \dim G_d V).$$

These equalities imply $\dim X_d > \dim G_d V$ if $q_Q(d) \leqslant 0$. In this case, there are infinitely many orbits, hence infinitely many isoclasses of indecomposables with dimension-function $\leqslant d$. The case arises when q_Q is not positive definite, because then there is a $d > 0$ such that $q_Q(d) \leqslant 0$.

If q_Q is positive definite and V indecomposable, our Lemma 7.2 implies

$$0 < q_Q(d) = 1 - (\dim X_d - \dim G_d V)$$

hence
$$q_Q(d) = 1 \text{ and } \dim X_d = \dim G_d V.$$

It follows that $G_d V$ is Zariski-open[7] and dense in X_d. Therefore, it coincides with the orbit of any other indecomposable in X_d, and the map $V \mapsto \underline{\dim}\, V$ provides an injection from the set of isoclasses of indecomposables into the set of positive roots.

It remains to prove that each positive root d is the dimension-function of an indecomposable: We already know that the number of isoclasses of indecomposables is finite. It follows that X_d contains only finitely many orbits, and one of them, say $G_d V$, must have the same dimension as X_d. So we have $1 = q_Q(d) = \dim \mathrm{Hom}(V, V)$, $\mathrm{Hom}(V, V) = k\mathbb{1}_V$ is local, and V is indecomposable. $\surd$

7.4. Let us return to the *general* case of a *finite quiver* Q. The objective is to describe the subset of $\mathbb{Z}^{Q_v}$ formed by the dimension-functions of the indecomposable representations. For this we consider the bilinear form $q_Q(d|e) = q_Q(d + e) - q_Q(d) - q_Q(e)$ associated with q_Q. By 7.2, this form satisfies

$$q_Q(\underline{\dim}\, V | \underline{\dim}\, W) = \dim \mathrm{Hom}(V, W) + \dim \mathrm{Hom}(W, V) - \dim \mathrm{Ext}^1(V, W)$$
$$- \dim \mathrm{Ext}^1(W, V).$$

It is also determined by the following formulas, where $e^i(i) = 1$ and $e^i(j) = 0$ if $j \neq i \in Q_v$:

$$\tfrac{1}{2} q_Q(e^i | e^i) = 1 - \text{number of loops } \overset{i}{\cdot}\!\circlearrowleft$$
$$-q_Q(e^i | e^j) = \text{number of arrows between } i \text{ and } j \neq i.$$

In particular, we have $q_Q(e^i | e^i) = 2$ if e^i is a *simple root*, i.e. if there is no loop at i. The formula

$$\sigma_i(d) = d - q_Q(e^i | d) e^i, \qquad d \in \mathbb{Z}^{Q_v},$$

then defines the *reflection* in the direction e^i, i.e. the automorphism of $\mathbb{Z}^{Q_v}$ which maps e^i onto $-e^i$ and fixes the vectors orthogonal to e^i. The group generated by these reflections is the *Weyl group* W_Q. The positive functions belonging to the orbit $W_Q e^i$ of a simple root e^i are the *real roots* (6.5). We denote their set by R_Q^{re}.

The *fundamental cone* $K_Q \subset \mathbb{Z}^{Q_v}$ consists of the positive functions d which satisfy $q_Q(e^i | d) \leqslant 0$ for each simple root e^i and have a connected (non-empty) *support*. Under the action of W_Q it generates the set $R_Q^{im} = \bigcup_{w \in W_Q} w K_Q$ of *imaginary roots.*[8]

Theorem[9]. *If d is a real root, there is exactly one isoclass of indecomposables with dimension-function d. If d is an imaginary root, there are infinitely many such isoclasses. There is none if $d \notin R_Q^{re} \cup R_Q^{im}$.*

If q_Q is positive definite, there is no imaginary root, and the real roots coincide with the positive roots as follows from our theorems or from 6.5.

If Q is an extended Dynkin quiver, the quadratic form q_Q is positive semi-definite. The *isotropic* functions, on which q_Q vanishes, are then integral multiples of the *isotropic generator* δ^Q (6.3). In this case, we have $R_Q^{im} = K_Q = \{n\delta^Q : n \in \mathbb{N} \setminus \{0\}\}$, and it is easy[10] to exhibit an infinite family of non-isomorphic indecomposables with dimension-function $n\delta^Q$, $n > 0$. In the case of Example 1 below, the required family is

$$\begin{tikzcd} k^n \arrow[rr, dash, "1_n"] \arrow[dr, "1_n"'] & & k^n \\ & k^n \arrow[ur, "s1_n + J_n"'] & \end{tikzcd}$$

where $s\mathbb{1}_n + J_n$ denotes a "Jordan-block" with eigenvalue s (1.7).

If Q contains a component which is neither Dynkin nor extended Dynkin, there are functions $d \in K_Q$ such that $q_Q(d) < 0$, but *there is no*[11] *positive d with support Q such that $q_Q(e^i | d) \geqslant 0$ for all $i \in Q_v$.*

Example 1. $Q\;\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c \arrow[ur] & \end{tikzcd}$, $q_Q(xe^a + ye^b + ze^c) = x^2 + y^2 + z^2 - yz - xz - xy$

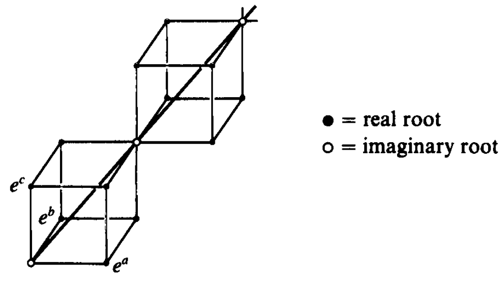

Fig. 3