III Geometry of Grassmannians

We turn now to geometric matters. Our objects of study will be, for the most part, *complex manifolds*. By this we mean they are spaces obtained by glueing together complex discs with holomorphic maps. For example, a 1-dimensional compact complex manifold is simply a Riemann surface; (or a nonsingular projective algebraic curve in the language of algebraic geometry). If the reader is so inclined, it is possible to simply ignore the complex structure and think of these objects as ordinary manifolds of twice the stated dimension. On the other hand, our spaces actually turn out to be projective, algebraic varieties over $\mathbb{C}$ so the machinery of algebraic geometry can profitably be brought to bear; (more on this in §5).

The prototypical complex manifold is the *complex projective space* $\mathbb{CP}^n$. It is obtained from $\mathbb{C}^{n+1} - \{0\}$ by making the identifications $(z_0,\ldots,z_{n+1}) \sim \lambda(z_0,\ldots,z_{n+1})$ where λ is a non-zero complex number. This yields the homogeneous coordinates $[z_0,\ldots,z_n]$ so, for example, the open set $\{z_0 \neq 0\}$ gives a typical coordinate patch under the map $[z_0,\ldots,z_n] \to (z_0^{-1}z_1,\ldots, z_0^{-1}z_n)$. By associating to a point $(z_0,\ldots,z_{n+1}) \in \mathbb{C}^{n+1} - \{0\}$ the line passing through it and the origin, we can view $\mathbb{CP}^n$ as the space of lines in $\mathbb{C}^{n+1}$. (This is equivalent to compactifying $\mathbb{C}^n$ by adding an ideal point at infinity for each line in $\mathbb{C}^n$).

The protagonist of this chapter is a mild generalization of $\mathbb{CP}^n$. We let $G_k(\mathbb{C}^{n+k})$ denote the space of k-dimensional linear subspaces of $\mathbb{C}^{n+k}$; it is called a Grassmann manifold or *Grassmannian* (see §1). Clearly, $G_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$. For projective geometers, $G_k(\mathbb{C}^{n+k})$ can be viewed as the space of $\mathbb{CP}^{k-1}$'s in $\mathbb{CP}^{n+k-1}$, in the obvious way.

In part, we hope to explain here the ubiquity of Grassmannians. They play a decisive role in the study of complex vector bundles (§2) and, in general, serve as an interesting and tractable class of examples for geometers and topologists. The geometry of these spaces is faithfully reflected in their cohomology. There are (at least) three different approaches to studying the cohomology of $G_k(\mathbb{C}^{n+k})$; we work with complex coefficients.

In section 2, we consider the Grassmannian as a universal example (or classifying space) for complex vector bundles. We associate algebraic invariants to such bundles called Chern classes and, following Borel [12], compute the cohomology of $G_k(\mathbb{C}^{n+k})$ in terms of these Chern classes.

Section 3 exhibits a cell-decomposition of the Grassmannian and hence an additive basis for the cohomology. This approach can boast the longest history going back to the enumerative geometry of Schubert [124]. We follow Chern's presentation [24]. In particular, we give geometric proofs of the Pieri formula and the Giambelli formula, the cornerstones of the Schubert calculus.

The flag manifold and its cohomology is studied in section 4 with the help of the Bruhat decomposition. In this way, the coinvariant algebra S_W (III, §3) makes a reappearance.

We conclude in section 5 with a brief description of several other approaches to interpreting the cohomology of $G_k(\mathbb{C}^{n+k})$.

§1. PRELIMINARIES

We begin with some basic facts about the Grassmannian $G_k(\mathbb{C}^{n+k})$.

Classically, these manifolds arose in differential geometry as a suitable target for a Gauss map. By this we mean, if M is a smooth, complex manifold of dimension k embedded in $\mathbb{C}^{n+k}$, then the assignment $x \mapsto T_xM$(= the tangent space of M at $x \in M$) defines a continuous map $M \to G_k(\mathbb{C}^{n+k})$. This is a special case of a classifying map that we will discuss in §2.

How do we topologize the set $G_k(\mathbb{C}^{n+k})$? Let $V_k(\mathbb{C}^{n+k})$ (resp. $\widetilde{V}_k(\mathbb{C}^{n+k})$) denote the set of all k-tuples $(x_1,\ldots,x_k) \in (\mathbb{C}^{n+k})^k$ such that the entries are linearly independent (resp. orthonormal). $V_k(\mathbb{C}^{n+k})$ inherits the subspace topology from $\mathbb{C}^{nk+k^2}$ and is called a *Stiefel manifold*. There is an obvious projection $p:V_k(\mathbb{C}^{n+k}) \to G_k(\mathbb{C}^{n+k})$ defined by mapping a k-tuple to the span of its entries. We give the set $G_k(\mathbb{C}^{n+k})$ the quotient topology and call it the *Grassmannian*. Since $G_k(\mathbb{C}^{n+k})$ is also a quotient of the compact space $\widetilde{V}_k(\mathbb{C}^{n+k})$, the Grassmannian is compact. Before proceeding further, we record

(1.1) *Lemma*. $G_k(\mathbb{C}^{n+k})$ is a compact, complex manifold of dimension nk.

Proof. To show $G_k(\mathbb{C}^{n+k})$ is Hausdorff, we separate points by a continuous function. If $x \in \mathbb{C}^{n+k}$, $X \in G_k(\mathbb{C}^{n+k})$, let $d_x(X)$ denote the squared distance from x to X. If $X \neq X' \in G_k(\mathbb{C}^{n+k})$, pick $x \in X - X'$. Then $d_x(X) = 0$, $d_x(X') \neq 0$ and the assertion is proven.

Now suppose $X_0 \in G_k(\mathbb{C}^{n+k})$ and we want to find a neighborhood U analytically isomorphic to $\mathbb{C}^{nk}$. We let $U = \{Y \in G_k(\mathbb{C}^{n+k}): Y \cap X_0^\perp = 0\}$. Each $Y \in U$ can be thought as the graph of a function $f_Y:X_0 \to X_0^\perp$. This map $Y \mapsto f_Y$ establishes an analytic isomorphism $U \approx \mathrm{Hom}(X_0,X_0^\perp) \approx \mathbb{C}^{kn}$.

We will usually assume $k \le n$, since there is a natural identification $G_k(\mathbb{C}^{n+k}) \approx G_n(\mathbb{C}^{n+k})$ given by $X \to X^\perp$ (orthogonal complement).

There is a canonical embedding of $G_k(\mathbb{C}^{n+k})$ into complex projective space $\mathbb{CP}^N$, where $N = \binom{n+k}{k} - 1$. In particular, $G_k(\mathbb{C}^{n+k})$ is a projective, complex algebraic variety. We can describe this *Plücker embedding* π explicitly.

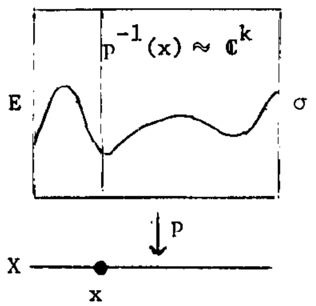
If $X \in G_k(\mathbb{C}^{n+k})$, then $\Lambda^k X$ is a line in $\Lambda^k\mathbb{C}^{n+k}$, hence a point $\pi(X)$ in $\mathbb{P}(\Lambda^k\mathbb{C}^{n+k}) = \mathbb{CP}^N$. Hence, if X has a basis $x_1,\ldots,x_k$, X is sent to $[x_1 \wedge\ldots\wedge x_k]$. It is not hard to check that π is an embedding. Furthermore, the image $\pi(G_k(\mathbb{C}^{n+k}))$ is the locus of a system of $\binom{n+k}{k+1}$ quadratic equations in the homogeneous coordinates of $\mathbb{CP}^N$ (namely the equations that pick out the *decomposable* forms). In particular, the Grassmann manifold $G_2(\mathbb{C}^4)$ yields (a very famous) non-singular quadric hypersurface in $\mathbb{CP}^5$. These quadratic relations were first discovered by Grassmann in 1844. For more on this fascinating geometry, see Griffiths and Harris [59 , Ch. 6].

§2. CHERN CLASSES

If M is a complex manifold and T_pM denotes its tangent space at $p \in M$, then we can "glue" all these complex vector spaces together into a new manifold: the *tangent manifold* TM. There is a projection map $p:TM \to M$ that tells you where the tangent vector starts. This is the canonical example of a complex vector bundle. Formally

(2.1) *Definition*. A *complex vector bundle* ξ of *dimension* k over a space X is a continuous surjection $p:E \to X$ whose *fibers* $p^{-1}(x)$, $x \in X$, have the structure of a complex k-dimensional vector space and satisfies the local triviality condition; i.e. every $x \in X$ has a neighborhood U and a homeomorphism $h:U \times \mathbb{C}^k \to p^{-1}(U)$ (*local coordinate chart*) which induces a linear isomorphism $\{x\} \times \mathbb{C}^k \to p^{-1}(x)$ for each $x \in U$. A *section* of ξ is a map $\sigma:X \to E$ satisfying $p\circ\sigma = 1_X$.

A vector bundle (with a typical section σ) can be pictured as



We refer to X as the *base space* and E as the *total space* of ξ. A vector bundle is *trivial* if there is a global coordinate chart, i.e. $E = X \times \mathbb{C}^k$. A vector bundle of dimension 1 is called a *line bundle*. Notice that a section σ to the tangent bundle $p:TM \to M$ is precisely a *vector field* on the manifold M. A good reference for all of this material is Husemoller [76, Ch. 3].

Let ξ and ξ' be complex vector bundles over X and X', respectively. A *map* $f:\xi \to \xi'$ is a commutative square

$$\begin{tikzcd} E \arrow[r, "f_1"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f_2"'] & X' \end{tikzcd}$$

such that the map on fibers

$$f_{1,x}: p^{-1}(x) \to p'^{-1}(f_2(x))$$

is linear. In addition, if $X = X'$, we insist $f_2 = 1_X$ and drop the subscript from f_1.

Suppose ξ and ξ' are both bundles over a space X and $f:\xi \to \xi'$ is a map. When is f an isomorphism? One necessary and sufficient condition is:

(2.2) *Lemma.* If $f:\xi \to \xi'$ is a bundle map over X

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

then f is an isomorphism if and only if f_x is a linear isomorphism for each $x \in X$.

Proof. One direction is obvious since an inverse bundle map $g:\xi' \to \xi$ produces linear inverses g_x for each f_x; $x \in X$. For the other direction, pick an open neighborhood U of x small enough so that there are local coordinate charts for both p and p'

$$U \times \mathbb{C}^k \xrightarrow{s} p^{-1}(U)$$

$$U \times \mathbb{C}^k \xrightarrow[t]{} p'^{-1}(U)$$

It is not hard to see that it only remains to show that the map $s^{-1} \circ f^{-1} \circ t$ is continuous where f^{-1} is obtained by glueing together all the f_x^{-1}'s. Now the map $t^{-1} \circ f \circ s$ is given by a continuous map of the form $(x,\alpha) \mapsto (x,\varphi_x(\alpha))$ where φ is a continuous map $X \to GL(\mathbb{C}^k)$. Hence the map $s^{-1} \circ f^{-1} \circ t$ is given by $(x,\alpha) \to (x,\varphi_x^{-1}(\alpha))$ which is also continuous since the operation of taking inverses is a continuous function of the matrix entries.

Essentially, any natural operation that can be performed on vector spaces can be extended functorially to vector bundles over the same base space. For example, direct sum, tensor product, exterior power, etc. are all legitimate methods of constructing new vector bundles out of old ones. Another technique is "pulling-back". Suppose ξ is a vector bundle over X and $f:Y \to X$ is a continuous map. Then the *pull-back bundle* $f^*(\xi)$ has total space

$$f^*E = Y \underset{X}{\times} E = \{(y,e) \in Y \times E : f(y) = p(e)\}$$

and projection $f^*(p)(y,e) = y$ onto the base space Y. We say

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

is a pull-back square. The top horizontal map is projected onto the second factor. We leave it as an exercise for the reader to check the local triviality condition. We now can invoke (2.2) to get a criterion for pull-backs.

(2.3) *Corollary*. Suppose ξ is a bundle over X, η a bundle over Y and $f:Y \to X$ a continuous map. Then $\eta \approx f^*(\xi)$ if and only if there is a bundle map $u:\eta \to \xi$ such that the map u_1 on total spaces gives a linear isomorphism on each fiber.

Proof. The necessity of the condition is easy to check. For the converse, we have the diagram

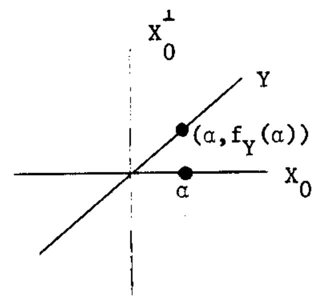
$$\begin{tikzcd} F \arrow[rr, "u_1"] \arrow[dr, dashed] \arrow[dd] & & E \arrow[dd] \\ & f^*E \arrow[ur] \arrow[dl] & \\ Y \arrow[rr, "u_2"'] & & X \end{tikzcd}$$

It suffices to show that the dotted map is an isomorphism and this follows from (2.2).

Our main goal is to show that every complex vector bundle over a compact space arises as the pull-back of a certain canonical bundle over the Grassmannian. We describe this bundle now. Define a total space $E_{k,n}$ consisting of pairs (V,x) where $x \in V$ and $V \in G_k(\mathbb{C}^{n+k})$. The projection map $p:(V,x) \mapsto V$ defines a map $E_{k,n} \to G_k(\mathbb{C}^{n+k})$ and a complex k-dimensional vector bundle over $G_k(\mathbb{C}^{n+k})$, called the *universal bundle* $\xi_{k,n}$. Since $G_k(\mathbb{C}^{n+k}) \approx G_n(\mathbb{C}^{n+k})$ there is also a bundle $\xi_{n,k}$ over $G_k(\mathbb{C}^{n+k})$ and the direct sum $\xi_{n,k} \oplus \xi_{k,n}$ is the trivial bundle. It remains to check:

(2.4) *Lemma*. The projection p is locally trivial.

Proof. Let $X_0 \in G_k(\mathbb{C}^{n+k})$ and U the open set containing it described in (1.1). We define a local coordinate chart $U \times X_0 \to p^{-1}(U)$ by $(Y,\alpha) \mapsto (Y,(\alpha,f_Y(\alpha)))$, in the notation of (1.1). The picture is:



where $\mathbb{C}^{n+k} = X_0 \oplus X_0^\perp$. Clearly this works.

If ξ is a k-dimensional bundle over X and f is a map satisfying $\xi = f^*(\xi_{k,n})$, for some large enough n, we will call f a *classifying map* for ξ. We will show that if X is compact, such maps always exist. The first reduction is

(2.5) *Proposition.* If E is the total space of ξ and $F:E \to \mathbb{C}^{n+k}$ is a continuous map that gives a linear isomorphism on each fiber, then there exists a classifying map for ξ.

Proof. We use F to construct a bundle map $\xi \xrightarrow{\psi} \xi_{k,n}$ by $\psi(e) =$ (F(fiber through e), F(e)). By the assumption on F and (2.3) we get $\xi \approx \psi^*(\xi_{k,n})$.

Hence it remains to prove

(2.6) *Theorem.* If ξ is a complex vector bundle of dimension k over a compact space X, then there is a map $F:E \to \mathbb{C}^{n+k}$, for n large enough, as in (2.5).

Proof. By compactness, choose a finite cover $U_1,\ldots,U_m$ with local coordinate charts $h_i:U_i \times \mathbb{C}^k \to p^{-1}(U_i)$. Choose a partition of unity $\{\eta_i\}$ subordinate to the cover $\{U_i\}$ with $\eta_i^{-1}(0,1] \subseteq U_i$. Define

$$g = \sum_i g_i : E \to \sum_i \mathbb{C}^k$$

by $g_i|p^{-1}(U_i) = (\eta_i p)\circ(p_2 h_i^{-1})$ (where $p_2:U_i \times \mathbb{C}^k \to \mathbb{C}^k$ is projection on the second factor) and zero otherwise. We leave it to the reader to check that g satisfies the condition of (2.5).

Hence, we can now conclude

(2.7) *Theorem.* Every k-dimensional complex vector bundle ξ over a compact space X is of the form $f^*(\xi_{k,n})$ for some map $f:X \to G_k(\mathbb{C}^{n+k})$, for n large enough.

Proof. Combine (2.5) and (2.6).

In particular, when ξ is the complex tangent bundle of a complex manifold M, the classifying map f is the Gauss map of an embedding of f in $\mathbb{C}^N$, N large enough. Clearly, (2.7) indicates the decisive role that the Grassmannian plays as a "universal" space in the theory of vector bundles. Furthermore, one can prove that $f^*(\xi_{k,n})$ is isomorphic to $g^*(\xi_{k,n})$ if and only if f and g are homotopic maps (see Husemoller [76 , p. 29-32]). These are the beginnings of topological K-theory.

How does the theory of complex vector bundles produce cohomological information about $G_k(\mathbb{C}^{n+k})$? In general, if ξ is a complex vector bundle of dimension k we can associate to it cohomological invariants called *Chern classes* $c_i(\xi) \in H^{2i}(B;\mathbb{Z})$, $1 \le i \le k$. These classes satisfy:

(C1) (Pull-backs) If ξ is a vector bundle over X and $f:Y \to X$ is a continuous map, then

$$c_i(f^*(\xi)) = f^*(c_i(\xi))$$

(C2) (Whitney sum) If $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots \in H^*(B;Z)$ is the total Chern class of ξ then

$$c(\xi \oplus \eta) = c(\xi)\cdot c(\eta)$$

(where the multiplication on the right is cup-product).

(C3) (Non-triviality) If $\xi_{1,n}$ is the canonical bundle over $\mathbb{CP}^n$ then

$$0 \neq c_1(\xi_{1,n}) \in H^2(\mathbb{CP}^n;\mathbb{Z})$$

Indeed, these axioms characterize the Chern classes. This is simple to prove once one knows the following easy consequence of the Projective Bundle theorem [62 ,p. 430].

(2.8) *Theorem*. (Splitting). If ξ is a complex vector bundle over a compact space X, then there is a map $f:Y \to X$ so that

(a) $f^*:H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is a monomorphism and

(b) $f^*(\xi)$ is a sum of line bundles.

Now the argument for the uniqueness of the Chern classes goes like this. Use (2.8), (C1) and (C2) to reduce the question to one concerning line bundles. If λ is a line bundle, use (2.7) to choose a classifying $f:X \to G_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$. But then $c_1(\lambda) = c_1(f^*(\xi_{1,n})) = f^*(c_1(\xi_{1,n}))$, so it is determined by (C3).

Of course, it still remains to construct classes $c_i(\xi)$ satisfying the axioms. This can be done in an abstract way using the Projective Bundle theorem [62 , pp. 429]. Historically though, these classes arose as obstructions to extending a section of the vector bundle over larger skeleta. In the tangent bundle situation, this amounts to constructing a vector field on the manifold.

Let us write c_i for $c_i(\xi_{k,n})$, $1 \le i \le k$, and $\bar{c}_j$ for $c_j(\xi_{n,k})$, $1 \le j \le n$. Recall $\xi_{k,n} \oplus \xi_{n,k} = 1$, so by (C2) we get

$$(*) \qquad (1 + c_1 + \cdots + c_k)(1 + \bar{c}_1 + \cdots + \bar{c}_n) = 1$$

Furthermore, Borel [12] has proven that these are the only relations; namely

$$H^*(G_k(\mathbb{C}^{n+k});\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_k,\bar{c}_1,\ldots,\bar{c}_n]/I_{n,k}$$

where $I_{n,k}$ is the ideal generated by the homogeneous parts of (*). We call this the *Borel picture* of the cohomology. Indeed, the first n homogeneous parts of (*) allow one to solve for $\bar{c}_j$ as a polynomial in the $c_1,\ldots,c_n$; e.g. $\bar{c}_2 = c_1^2 - c_2$. Suppose now that the remaining k relations among the

c_i's are $f_{1,n},\ldots,f_{k,n}$. Then it is shown in [65 , App.] that these relations are given by the first column of

$$\begin{pmatrix} c_1 & 1 & & 0 \\ c_2 & & \ddots & \\ \vdots & & & 1 \\ c_k & 0 & & 0 \end{pmatrix}^{n+1}$$

Hence we can write $H^*(G_k(\mathbb{C}^{n+k});\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_k]/(f_{1,n},\ldots,f_{k,n})$. (The proof is via high-school algebra). The reader should write down some examples with $k = 2$, n small, to get a feeling for these relations.

In §4 we will obtain a different way of thinking about Borel's computation

§3. SCHUBERT CALCULUS

The Schubert calculus was invented a century ago by H. Schubert and canonized in his book "Kalkül der Abzählenden Geometrie" published in 1879. The subject of enumerative geometry that Schubert pioneered is concerned with counting various geometric configurations in $\mathbb{CP}^3$, and only later in 1886 did he systematically consider questions in higher dimensions. For example, Schubert claims that there are 666,841,048 quadric surfaces tangent to 9 given quadric surfaces in $\mathbb{CP}^3$. According to Kleiman [85], it is not clear whether this number or the method it is based on are completely sound. Indeed, cleaning up such confusion is the intent of Hilbert's 15th Problem. (For more on Schubert's work, see the fine surveys [84], [85] and [59, Ch. 6]

Fortunately, the foundations of the Schubert calculus have been secured for problems involving linear subspaces. The first such attempt at a rigorour treatment of the linear Schubert program was suggested by van der Waerden [154] and proceeded by exploiting the recent intersection theory of Lefschetz.

This was then done in a systematic geometric way by Ehresmann [44] by constructing explicit cell-decompositions of many complex algebraic varieties, including the Grassmann manifold. Further information concerning the mutual disposition of the cells was provided by the work of Hodge [70] and Todd.

We will take the algebraic point of view that a Schubert calculus for a graded algebra H^*, in general, consists of three basic ingredients:

(3.0 i) *Basis Theorem*. A linear basis of *Schubert classes* $\{Z_w\}$ for H^* and a collection of classes $\{Y_s\}$ (called *special classes*), that algebraically generate H^*.

(3.0 ii) *Pieri formula*. A formula that describes the multiplication of Schubert classes by special classes:

$$Z_w \cdot Y_s = \sum_{w'} c_{w,s}^{w'} Z_{w'}$$

(3.0 iii) *Giambelli formula*. A formula that expresses each Schubert class. Z_w as a polynomial in the special classes $\{Y_s\}$.

We begin with the easiest example. There is a simple geometric cell-decomposition of the projective spaces $\mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$. Namely, let

$$0 < V_1 < \cdots < V_n < \mathbb{C}^{n+1}$$

be a fixed flag and define

$$\langle i\rangle = \{\ell \in \mathbb{C}P^n : \ell \subset V_{i+1}, \ell \not\subset V_i\}.$$

Each $\langle i\rangle$ is homeomorphic to an open $2i$-cell e^{2i}. For example,

$$\begin{aligned}\langle 1\rangle &= \{\ell \in \mathbb{C}P^n : \ell \subset V_2, \ell \not\subset V_1\} \\ &= \{\ell \subset V_2,\ \ell \neq V_1\} \\ &= \mathbb{C}P^1 - \{V_1\} \approx S^2 - \{V_1\} \approx \mathbb{C} \approx e^2 .\end{aligned}$$

As is well-known, we can take the $\langle i\rangle$'s for the Schubert classes and $\langle 1\rangle$ as the single special class.

We will mimick this construction for the Grassmann manifold $G_k(\mathbb{C}^{n+k})$. The only real difference is that one requires a slightly more complicated scheme of bookkeeping. The basic combinatorial object is a sequence of integers $0 \le a_1 \le \cdots \le a_k \le n$. We then define:

$$\langle a_1,\ldots,a_k\rangle = \{X \in G_k(\mathbb{C}^{n+k}) : \dim(X \cap V_{a_i+i}) = i\}$$

where, as above, $0 < V_1 < \cdots < V_{n+k-1} < \mathbb{C}^{n+k}$ is a fixed reference flag. Suppose $X \in G_k(\mathbb{C}^{n+k})$. Then consider the intersections

$$0 < X \cap V_1 < X \cap V_2 < \cdots < X \cap V_{n+k-1} < X$$

Clearly, each step in this chain can increase the dimension by at most one and there are exactly k "jumps" in order to get all of X. Let $d_1,\ldots,d_k$ be these jump points i.e.

$$\dim(X \cap \mathbb{C}^{d_i-1}) = i - 1 \quad \dim(X \cap \mathbb{C}^{d_i}) = i.$$

Then we will write $X \in \langle d_1-1,\ldots,d_k-k\rangle$. (We could just as easily have used a notation based on the d_i's, as we will do later, but for the moment the modified sequence of a_i's, $a_i = d_i - i$, is more convenient).

We claim now that the set $\langle a_1,\ldots,a_k\rangle$ is homeomorphic to a cell of dimension $2\sum a_i$. We proceed by representing a k-plane X in $\langle a_1,\ldots,a_k\rangle$ by a $k \times (n+k)$ matrix, the rows of which span X.

Firstly, $\dim(X \cap \mathbb{C}^{a_1+1}) = 1$, so we can pick a $v_1 \in X \cap \mathbb{C}^{a_1+1}$ and insist $v_1 \cdot e_{a_1+1} = 1$ to make it unique. Then, we choose $v_2 \in X \cap \mathbb{C}^{a_2+2}$

satisfying $v_2 \cdot e_{a_2+2} = 1$ and $v_2 \cdot e_{a_1+1} = 0$. Continuing in this fashion we get a matrix of the form

$$\begin{pmatrix} * \cdots 1 & 0 & & & \cdots & & 0 \\ * \cdots 0 & * \cdots * & 1 & 0 & & \cdots & 0 \\ * \cdots 0 & * \cdots * & 0 & * \cdots * & 1 & 0 \cdots & 0 \\ \vdots & & & & & & \\ \vdots & & & & & & \end{pmatrix}$$

where the (i,j)th entry is zero, for $j > a_i + i$. It only remains to count how many $*$'s appear to determine the dimension. The i^{th} row contains a_i so we get $\langle a_1,\ldots,a_k\rangle$ is complex $(\sum_{i=1}^{k} a_i)$-space.

It is possible to show that the closure $[a_1,\ldots,a_k] = \overline{\langle a_1,\ldots,a_k\rangle}$ is the union of all "smaller" $\langle b_1,\ldots,b_k\rangle$, i.e. $[a_1,\ldots,a_k] = \bigcup \langle b_1,\ldots,b_k\rangle$ where $\langle b_1,\ldots,b_k\rangle$ satisfies $b_i \le a_i$, for all i, $1 \le i \le k$. Hence

$$[a_1,\ldots,a_k] = \{X \in G_k(\mathbb{C}^{n+k}) : \dim(X \cap V_{a_i+i}) \ge i\}$$

Each $[a_1,\ldots,a_k]$ determines a homology class in $G_k(\mathbb{C}^{n+k})$ and we have

(3.1) *Basis Theorem.* The integral homology $H_*(G_k(\mathbb{C}^{n+k}),\mathbb{Z})$ is freely generated by the homology classes $[a_1,\ldots,a_k]$ of dimension $2\sum_{i=1}^{k} a_i$.

Proof. We need only observe that since all these chains are even-dimensional they are all cycles and there are no boundaries to worry over.

We now come to a few remarks.

(3.2) The class $[a_1,\ldots,a_k]$ does not depend on the choice of the reference flag, since all flags can be moved to the standard one by a continuous automorphism of $\mathbb{C}^{n+k}$.

(3.3) We let $(a_1,\ldots,a_k) \in H^{2\Sigma a_i}(G_k(\mathbb{C}^{n+k});\mathbb{Z}) \approx \mathrm{Hom}(H_{2\Sigma a_i}(G_k(\mathbb{C}^{n+k}),\mathbb{Z})$ assign 1 to $[a_1,\ldots,a_k]$ and zero otherwise. (We are, of course, using the freeness assertion of (3.1)). Then the Chern classes and the normal Chern classes can be identified as specific *Schubert classes* $(a_1,\ldots,a_k)$. Namely,

$$c_i = (0,\ldots,0,\underbrace{1,\ldots,1}_{i}) \qquad 1 \le i \le k$$

$$\bar{c}_j = (-1)^j(0,\ldots,0,j) \qquad 1 \le j \le n.$$

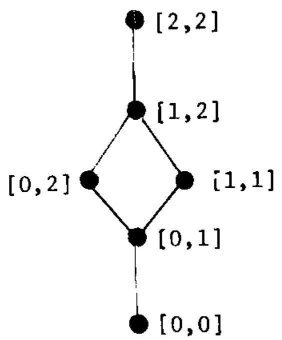
The normal Chern classes $\bar{c}_j$ play an important role in the Schubert calculus and so are often called *special* Schubert classes (cf. (3.0 i) and (3.7)).

(3.4) There is an intersection pairing

$$H_{2nk-x_1}(G_k(\mathbb{C}^{n+k})) \times H_{2nk-x_2}(G_k(\mathbb{C}^{n+k})) \to H_{2nk-(x_1+x_2)}(G_k(\mathbb{C}^{n+k}))$$

where we write $\alpha \cdot \beta$ (the $\mathbb{Z}$ coefficients are assumed). It is dual to the ordinary cup-product in the cohomology ring $H^*(G_k(\mathbb{C}^{n+k});\mathbb{Z})$.

(3.5) Finally, let us take a concrete look at the classes $[a_1,\ldots,a_k]$ in a simple example. The first case that is not a projective space is $k = n = 2$, the space $G_2(\mathbb{C}^4)$ of complex 2-planes in $\mathbb{C}^4$. The Hasse diagram of the partial order of Schubert classes is:



Clearly, $[2,2] = G_2(\mathbb{C}^4)$, and $[0,0]$ is a point determined by the choice of V_2 in the standard flag

$$0 < V_1 < V_2 < V_3 < \mathbb{C}^4$$

Similarly,

$$[1,2] = \{X:\dim(X \cap V_2) \geq 1\}$$
$$[1,1] = \{X:X \subset V_3\}$$
$$[0,2] = \{X:V_1 \subset X\}$$
$$[0,1] = \{X:V_1 \subset X \subset V_3\}$$

The main goal of this section is to give a geometric description of the multiplicative structure of $H^*(G_k(\mathbb{C}^{n+k});\mathbb{Z})$. The dual of the Basis Theorem gives us additive generators $(a_1,\ldots,a_k)$ and it suffices to compute the coefficients in a linear expression for $(a_1,\ldots,a_k) \cup (b_1,\ldots,b_k)$.

It suffices to prove the following two results (cf. (3.0 i, ii)).

(3.6) *Pieri formula.* $\displaystyle (a_1,\ldots,a_k)\cdot\bar{c}_j = \sum_{\substack{a_i \leq b_i \leq a_{i+1} \\ \Sigma b_i = j+\Sigma a_i}} (b_1,\ldots,b_k)$ [116]

(3.7) *Giambelli formula.* $(a_1,\ldots,a_k) = \det(\bar{c}_{a_i+(j-i)})$ with the convention that $\bar{c}_\ell = 0$, if ℓ is not between 0 and n. [57]

This is the *Schubert picture* of the cohomology. We begin with the following useful remark.

(3.8) *Proposition.* If $[a_1,\ldots,a_k]\cdot[b_1,\ldots,b_k] \neq 0$, then $b_i + a_{k-i+1} \geq n$.

Proof. If $b_i + a_{k-i+1} < n$, then

$$(b_i + i) + a_{k-i+1} + k-i+1 = b_i + a_{k-i+1} + k+1$$
$$< n+k+1$$

so we can find subspaces V, W of $\mathbb{C}^{n+k}$ of dimensions $b_i + i$, $a_{k-i+1} + k-i+1$ respectively so that $V \cap W = 0$. We extend these to flags $\{V_i\}$, $\{W_i\}$ for $[b_1,\ldots,b_k]$, $[a_1,\ldots,a_k]$ respectively. Now, if $X \in [a_1,\ldots,a_k]\cdot[b_1,\ldots,b_k]$ then

$$\dim(X \cap V) \geq i$$

and

$$\dim(X \cap W) \geq k - i + 1$$

so that $0 \neq (X \cap V) \cap (X \cap W) \subseteq V \cap W = 0$, a contradiction. This completes the proof.

This allows us to compute the intersection product for cycles in complementary dimensions.

(3.9) *Corollary.* If $\sum_{i=1}^{k} (a_i + b_i) = nk$ and $[a_1,\ldots,a_k]\cdot[b_1,\ldots,b_k] \neq 0$ then $b_i = n - a_{k-i+1}$. In this case, the product is 1.

Proof. By (3.7), $b_i + a_{k-i+1} \geq n$. Since summing over i on the left gives nk, each of these inequalities must be equalities. The last assertion is clear.

It is now possible to explicitly identify the Poincaré duality isomorphism for the Grassmann manifold.

(3.10) *Corollary.* The Poincaré map $H_{2\Sigma a_i}(G_k(\mathbb{C}^{n+k}),Z) \to H^{2nk-2\Sigma a_i}(G_k(\mathbb{C}^{n+k});\mathbb{Z})$ sends $[a_1,\ldots,a_k]$ to $(n-a_k,\ldots,n-a_1)$.

Proof. This follows from (3.9) and the observation that the Poincaré map is the adjoint to the intersection pairing

$$H_{2\Sigma a_i}(G_k(\mathbb{C}^{n+k})) \times H_{2nk-2\Sigma a_i}(G_k(\mathbb{C}^{n+k})) \to H_0(G_k(\mathbb{C}^{n+k})) \approx \mathbb{Z}.$$

Our approach to the Pieri formula will be to attack the dual statement in homology. By (3.10), (3.6) dualizes to

$$[n-a_k,\dots,n-a_1][n-j,n,\dots,n] = \sum[n-b_k,\dots,n-b_1]$$

where the b_i's are as in (3.6). So by (3.9), it suffices to show

(3.6') $$[b_1,\dots,b_k]\cdot[n-a_k,\dots,n-a_1]\cdot[n-j,n,\dots,n] = \begin{cases} 1 & \text{if the } b_i\text{'s are as in (3.6)} \\ 0 & \text{otherwise} \end{cases}$$

We first analyze the simple product: $[b_1,\dots,b_k]\cdot[n-a_k,\dots,n-a_1]$. The first result is

(3.11) *Lemma.* If $0 \neq [b_1,\dots,b_k]\cdot[n-a_k,\dots,n-a_1]$, then $a_i \le b_i$, for all i.

Proof. By (3.8), $b_i + (n-a_i) \ge n$, and we are done.

Following Chern [26], we pick convenient reference flags for the cells $[b_1,\dots,b_k]$ and $[n-a_k,\dots,n-a_1]$. These will be fixed for the remainder of the proof. We will write the coordinates of $\mathbb{C}^{n+k}$ as $x_1,\dots,x_k$, $y_1,\dots,y_n$. Let V_i be the subspace of dimension b_i+i given by the equations

$$x_1 = \dots = x_{k-i} = 0 = y_{b_i+1} = \dots = y_n = 0$$

and let W_i be the subspace of dimension $n-a_{k-i+1}+i$ given by

$$x_{i+1} = \dots = x_k = 0 = y_1 = \dots = y_{a_{k-i+1}}$$

We also define $M_i = V_i \cap W_{k-i+1}$, whose non-zero coordinates are easily checked to be $\{x_{k-i+1}, y_{a_i+1},\dots,y_{b_i}\}$. Hence M_i has dimension $b_i - a_i + 1$.

Recall, that a k-dimensional subspace X can be viewed as a certain $k \times (n+k)$ matrix of row rank k. Let $A = \mathbb{C}^k$ and view the column vectors $\xi_1,\dots,\xi_k$, $\eta_1,\dots,\eta_n$ as elements of A. Clearly they generate A, since column rank equals row rank. Using this idea we prove:

(3.12) *Proposition.* If $X \in [b_1,\dots,b_k]\cdot[n-a_k,\dots,n-a_1]$, then $X \subseteq M$, the span of the subspaces M_i.

Proof. By definition, $\dim(X \cap V_i) \ge i$ and $\dim(X \cap W_{k-i}) \ge k-i$. These facts imply $\xi_1,\dots,\xi_{k-i}$, $\eta_{b_i+1},\dots,\eta_n$ span a $k-i$ dimensional subspace of A and $\xi_{k-i+1},\dots,\xi_k$, $\eta_1,\dots,\eta_{a_i+1}$ span an i-dimensional subspace of A. If they intersect in A, then together they span a space of dimension $\le k-1$, a contradiction. So if $b_i < a_{i+1}$, then $\eta_{b_i+1} = \dots = \eta_{a_{i+1}} = 0$. So all the coordinates "outside" of M_i, (if any) are necessarily zero. This completes the proof.

We can now show that if the b_i's are not properly "interlaced" among the a_i's then the triple intersection product vanishes.

(3.13) *Lemma.* If $b_i > a_{i+1}$, for some i, then

$$[b_1,\dots,b_k]\cdot[a-a_k,\dots,n-a_1]\cdot[n-j,n,\dots,n] = 0.$$

Proof. By above remarks concerning the M_i's,

$$\dim(M) \le \sum_{i=1}^{k} 1 + (b_i - a_i) = k + j.$$

But the inequality $b_i < a_{i+1}$ implies that $M_i \cap M_{i+1} \neq 0$, so actually $\dim(M) < k+j$. Hence it is possible to choose a subspace H of $\mathbb{C}^{n+k}$ of dimension $n-j+1$ satisfying $H \cap M = 0$.

If we choose an $X \in [b_1,\ldots,b_k]\cdot[n-a_k,\ldots,n-a_1]$ then $X \subset M$, by (3.12). Hence $X \cap H \subset M \cap H = 0$. But it is easy to see that $X \in [n-j,n,\ldots,n]$ if and only if $\dim(X \cap H) \geq 1$. So we are done.

It only remains to prove

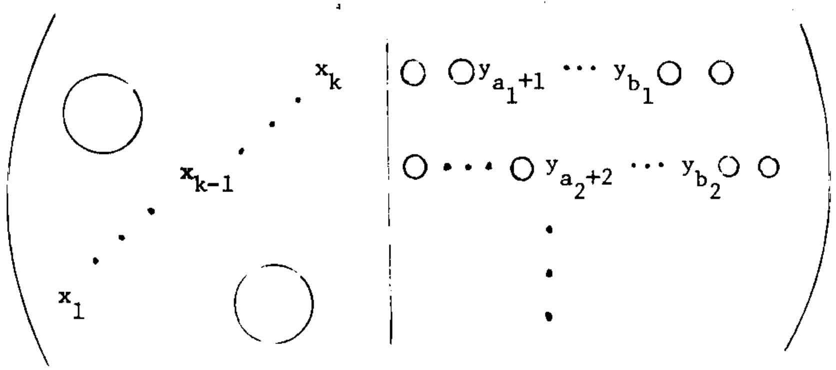
(3.14) *Lemma*. If $(b_1,\ldots,b_k)$ satisfies the conditions of (3.6) then $[b_1,\ldots,b_k]\cdot[n-a_k,\ldots,n-a_1]\cdot[n-j,n,\ldots,n] = 1$.

Proof. Now we have equality: $\dim M = k+j$, since $M_i \cap M_j = 0$, if $i \neq j$. Let H be the $(n-j+1)$-dimensional subspace given by the equations:

$$y_{a_i+1} = \cdots = y_{b_i} = 0 \qquad 1 \leq i \leq k$$

$$\frac{x_1}{\lambda_1} = \cdots = \frac{x_k}{\lambda_k}$$

for some choice of $\lambda_i \neq 0$. Clearly, $H \cap M$ is the line $\frac{x_1}{\lambda_1} = \cdots = \frac{x_k}{\lambda_k}$, since all the y-coordinates vanish. Now, if $X \in [b_1,\ldots,b_k]\cdot[n-a_k,\ldots,n-a_1]$ it is the usual argument to see $\dim(X \cap M_i) \geq 1$, so we can choose vectors $v_i \in X \cap M_i$, $1 \leq i \leq k$, to serve as a basis for X. The matrix for X will look like



If $X \in [b_1,\ldots,b_k]\cdot[n-a_k,\ldots,n-a_1]\cdot[n-j,n,\ldots,n]$, then $\dim(X \cap H) \geq 1$ but since $X \cap H$ is contained in the line $M \cap H$, it must actually be that line.

In particular, $(\lambda_1,\ldots,\lambda_k,0,\ldots,0) \in X$. So there exist numbers $\mu_1,\ldots,\mu_k \in \mathbb{C}$ such that

$$\lambda_i = \mu_i x_i \qquad 1 \leq i \leq k$$

$$0 = \mu_i y_k \qquad a_i + 1 \leq k \leq b_i \quad 1 \leq i \leq k$$

Hence, all the y_k's are zero and we conclude X can only be the space generated by the first n coordinate vectors.

This completes the proof of the Pieri formula. Somewhat curiously, it turns out that the Giambelli formula (3.7) is a consequence of the Pieri formula.

Proof of (3.7) . Let Δ denote the right-hand side. Expand the determinant Δ along the first column. Hence

$$\Delta = \sum_{i=1}^{k} (-1)^{i+1}\bar{c}_{a_i+i-1}(a_1-1,\ldots,a_{i-1}-1,\ a_{i+1},\ldots,a_k)$$

Applying (3.6) to each summand almost every term cancels excepts the desired one $(a_1,\ldots,a_k)$. (The reader is urged to check this in detail for himself).

Example. (a) We conclude this section with some typical computations in the Schubert calculus. Suppose we are working in the (real) 18-dimensional manifold $G_3(\mathbb{C}^6)$ and want to square $(0,1,2) \in H^6$. First we apply (3.7) and get

$$(0,1,2) = \det\begin{pmatrix} 1 & 0 & 0 \\ \bar{c}_2 & \bar{c}_1 & 1 \\ 0 & \bar{c}_3 & \bar{c}_2 \end{pmatrix} = \bar{c}_2\bar{c}_1 - \bar{c}_3$$

Then we invoke (3.6) and compute

$$(0,1,2)\bar{c}_1 = (1,1,2) + (0,2,2) + (0,1,3)$$
$$(1,1,2)\bar{c}_2 = (1,2,3)$$
$$(0,2,2)\bar{c}_2 = (2,2,2) + (1,2,3)$$
$$(0,1,3)\bar{c}_2 = (1,2,3) + (0,3,3)$$
$$(0,1,2)\bar{c}_3 = (1,2,3)$$

Hence, $(0,1,2)^2 = 2(1,2,3) + (2,2,2) + (0,3,3)$.

(b) Suppose we consider now $G_2(\mathbb{C}^4)$ as in (3.5). We think projectively for the moment and view this space as the set of complex $\mathbb{CP}^1$'s in $\mathbb{CP}^3$. The enumerative geometry of Schubert interprets the four-fold product of $(0,1)$ as counting the numbers of lines that meet four lines in $\mathbb{CP}^3$ in general position. We can compute

$$(0,1)^2 = (0,2) + (1,1)$$
$$(0,1)^3 = 2(1,2)$$
$$(0,1)^4 = 2(2,2)$$

So the answer is 2. (See [84],[85],[59] for other geometric examples and also the latter for an alternative approach to the Pieri formula).

§4. FLAG MANIFOLDS

A sequence of subspaces of $\mathbb{C}^m$, $0 < V_1 < \cdots < V_{m-1} < \mathbb{C}^m$, where $\dim(V_i) = i$, is called a *(complete) flag*. The set of all such flags is denoted $\mathrm{Flag}(\mathbb{C}^m)$. We can equip this set with a topology in a manner similar to that used for the Grassmannian. We leave these details to the reader. Similarly, one can also show $\mathrm{Flag}(\mathbb{C}^m)$ is a compact, complex manifold of dimension $\frac{m(m-1)}{2}$. The dimension assertion follows, for example, by induction and the existence of an obvious fibration

$$\mathrm{Flag}(\mathbb{C}^{m-1}) \longrightarrow \mathrm{Flag}(\mathbb{C}^m) \xrightarrow{p_{m-1}} G_{m-1}(\mathbb{C}^m) \cong \mathbb{CP}^{m-1}$$

Indeed, the map p_{m-1} is a special case of the projection maps

$$\mathrm{Flag}(\mathbb{C}^{n+k}) \xrightarrow{p_k} G_k(\mathbb{C}^{n+k})$$

where $p_k : (V_1 < \cdots < V_{m-1}) \mapsto V_k$. Cohomologically, the map p is very well-behaved. It induces an injection

$$p^* : H^*(G_k(\mathbb{C}^{n+k})) \to H^*(\mathrm{Flag}(\mathbb{C}^{n+k}))$$

and we will have more to say about this later.

As is the case with the cohomology of $G_k(\mathbb{C}^{n+k})$, the cohomology of $\mathrm{Flag}(\mathbb{C}^m)$ admits two different descriptions - a Borel picture (cf. §2) and a Schubert picture (cf. §3). We will consider both of them in this section.

It is convenient at this stage to introduce an alternative description of the flag manifold as a homogeneous space. There is a transitive action of $GL_m(\mathbb{C})$ on $\mathrm{Flag}(\mathbb{C}^m)$. Namely, if

$$0 < V_1 < \cdots < V_{m-1} < \mathbb{C}^m$$

is a flag and $g \in GL_m(\mathbb{C})$ then

$$g\cdot(0 < V_1 < \cdots < V_{m-1} < \mathbb{C}^m) = 0 < gV_1 < \cdots < gV_{m-1} < \mathbb{C}^m$$

The stabilizer of the standard flag is precisely the subgroup B of upper triangular matrices:

$$\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ \bigcirc & & * \end{pmatrix}$$

Hence we obtain an identification

$$GL_m(\mathbb{C})/B \approx \mathrm{Flag}(\mathbb{C}^m)$$

If instead we consider the action of the (maximal compact) unitary group $U_m(\mathbb{C})$, we get an identification

$$U_m(\mathbb{C})/T \approx \mathrm{Flag}(\mathbb{C}^m)$$

where T is the subgroup of diagonal matrices:

$$\begin{pmatrix} * & & \bigcirc \\ & \ddots & \\ \bigcirc & & * \end{pmatrix}$$

The advantage of the homogeneous space point of view is the natural extension of our problem to any complex semisimple Lie group G. Our B is replaced by a Borel subgroup, i.e. a maximal solvable connected subgroup, $U_m(\mathbb{C})$ by a maximal compact subgroup K and T by a maximal torus. In this general case there is a homeomorphism

$$K/T \approx G/B$$

We will refer to either of these spaces as a (generalized) *flag manifold*. We recall that the Weyl group W of K is defined as $N(T)/T$, where N denotes normalizer. The Borel picture of the cohomology of a flag manifold is summarized in the following omnibus result. This is the promised geometric interpretation of the coinvariant algebra.

(4.1) *Theorem*. (Borel) There is a map $c:S(V) \to H^*(G/B)$ (that multiplies degree by 2), where $V = \mathbb{C} \otimes_{\mathbb{Z}} X(T)$, the (complexified) character group

on T, that induces an isomorphism:

$$S_W = S(V)/I_W \approx H^*(G/B)$$

The map $\mathbb{C}$ is obtained on degree 1 by associating to a character χ of T the first Chern class of the corresponding line bundle L_χ over $K/T = G/B$.

On the other hand, there is also a Schubert-type cell-decomposition of the flag manifold. In the case of $G = GL_m$, we can work either in terms of $\mathrm{Flag}(\mathbb{C}^m)$ or G/B. We describe here both the geometric and algebraic picture in this special case and show how they are, in fact, identical.

We begin with the Schubert-type cells in $\mathrm{Flag}(\mathbb{C}^m)$. If X is a k-plane in $\mathbb{C}^m$, we will define (cf. §3) the *signature* of X to be $s(X) = (d_1 < \cdots < d_k)$ where the d_i are the "jump-points" for X, i.e.

$$X \cap \mathbb{C}^{d_i - 1} \neq X \cap \mathbb{C}^{d_i}$$

If $V_1 < \cdots < V_{m-1}$ is a flag we get a triangular array of signatures

$$\begin{aligned} s(V_1) &= d_{11} \\ s(V_2) &= d_{21} < d_{22} \\ s(V_3) &= d_{31} < d_{32} < d_{33} \qquad (4.1^*) \\ &\vdots \\ s(V_{m-1}) &= d_{31} < d_{m-1\,2} < \cdots < d_{m-1\,m-1} \end{aligned}$$

The following remark allows us to simplify this bulky notation.

(4.2) *Lemma*. If $V \subseteq V' \subseteq \mathbb{C}^m$, then $s(V) \subseteq s(V')$.

Proof. If $i \in s(V)$, we get a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & V \cap \mathbb{C}^{i-1} \arrow[r] \arrow[d, phantom, "\cap"] & V \cap \mathbb{C}^{i} \arrow[r, "i^{th}\text{ coordinate}"] \arrow[d, phantom, "\cap"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V' \cap \mathbb{C}^{i-1} \arrow[r] & V' \cap \mathbb{C}^{i} \arrow[r] & \mathbb{C} &
\end{tikzcd}$$

An easy diagram-chase shows the bottom sequence is also exact, so $i \in s(V')$.

Hence, as one goes down the rows in (4.1*) exactly one new number is added at each stage, so we get a sequence $d_{11}, d_{2i_2}, \ldots, d_{m-1\, i_{m-1}}$ of numbers between 1 and m. If we throw the remaining number at the end we get a permutation $w \in \Sigma_m$. We then define the *signature* σ of a flag $F = (V_1 < \cdots < V_{m-1})$ by $\sigma(F) = w$. Finally, in analogy to $G_k(\mathbb{C}^{n+k})$, we can define a *Bruhat cell* to be

$$B_w = \{F \in \mathrm{Flag}(\mathbb{C}^m) : \sigma(F) = w\}$$

The closure of such a cell is the *Schubert variety*

$$\bar{B}_w = \{F \in \mathrm{Flag}(\mathbb{C}^m) : \sigma(F) \le w\}$$

where $\le$ is interpreted as the *Bruhat order* (I, §6) on the Coxeter group Σ_m! In other words,

$$\bar{B}_w = \bigcup_{w' \le w} B_{w'}$$

The fundamental classes of the Schubert varieties $\bar{B}_w$ give homology classes X_w for G/B. These then yield Schubert cohomology classes $X_w \in H^*(\mathrm{Flag}(\mathbb{C}^m); \mathbb{Z})$ of dimension $2\ell(w)$, where ℓ is the length function on Σ_m!

In order to describe the Bruhat decomposition for G/B, G an arbitrary reductive group we need some notation. Fix a maximal torus T and a Borel

subgroup $B \supseteq T$. There is always another Borel subgroup B^- *opposite* to B satisfying $B \cap B^- = T$. If $G = GL_m(\mathbb{C})$ and B is the upper triangular matrices then B^- is the lower triangular matrices. There is also a decomposition of an arbitrary Borel $B = UT$ where U is the "unipotent" elements of B. If B is the upper triangular matrices, U consists of upper triangular matrices with 1's along the diagonal

$$\begin{pmatrix} 1 & * & & * \\ & 1 & \ddots & * \\ & 0 & & \ddots \\ & & & 1 \end{pmatrix}$$

We also write $B^- = U^- T$ and define

$$U_w^- = U \cap wU^-w^{-1}$$

The main algebraic result is (see [14, p. 347]).

(4.3) *Theorem. (Bruhat decomposition* [21]*)*. If G is a complex reductive Lie group and $T \subseteq B$, as above, then G is the disjoint union of double cosets BwB, $w \in W$. (This double coset is actually BnB, where $n \in N(T)$ represents w. This is well-defined since $T \subseteq B$). In addition, there is an isomorphism of algebraic varieties

$$U_w^- \times B \to BwB$$

Remark. Tits [153] has shown that the Bruhat decomposition is a formal consequence of the axioms for a BN-pair structure on a group. This is the most elegant and clean approach.

Dividing out by B and recalling $B = UT$ yields:

(4.4) *Corollary.* The homogeneous space G/B is a disjoint union of double cosets BwB/B, $w \in W$, and there are isomorphisms of algebraic varieties $BwB/B \approx U_w^- \approx \mathbb{C}^{\ell(w)}$.

We will show that $BwB/B = B_w$ under the identification $G/B \approx \mathrm{Flag}(\mathbb{C}^m)$. This will show that the Schubert varieties are identical since the closure in the Zariski or classical topology agree. We begin by proving the Bruhat decomposition in the special case $G = GL_m(\mathbb{C})$ in order to make the statement of (4.3) more concrete.

(4.5) *Proposition.* Every matrix $\alpha \in GL_m(\mathbb{C})$ can be written as a product $\beta_1 w \beta_2$, where β_i are upper triangular and w is a permutation matrix.

Proof. We can multiply the matrix α on the right by appropriate products of upper triangular unipotents U (elementary row operations) to maximize the number of zeros in each row. The number of zeros at the beginning of each row must be distinct by our maximal assertion. Now we can multiply by a permutation matrix P so that $\alpha UP = \beta \in B$; so $\alpha = \beta P^{-1} U^{-1}$ as claimed.

It remains to show:

(4.6) *Proposition.* If P_1, P_2 are permutation matrices and $BP_1B = BP_2B$ then $P_1 = P_2$.

Proof. We can choose $\beta, \beta' \in B$ such that

$$P_2 = \beta P_1 \beta'$$

so $P_1^{-1}\beta^{-1}P_2$ is upper triangular. If $\sigma_i \in \Sigma_m$ corresponds to P_i, $i = 1,2$, then $(P_1^{-1})_{i,j} = \delta_{\sigma_1^{-1}(i),j}$, $(P_2)_{i,j} = \delta_{\sigma_2(i),j}$ and, say,

$(\beta^{-1})_{i,j} = t_{i,j}$. Hence, if $i > j$

$$t_{\sigma_1^{-1}(i),\sigma_2^{-1}(j)} = 0$$

Hence $\sigma_1^{-1}(i) \neq \sigma_2^{-1}(j)$, for all $i > j$ (since the diagonal entries of U^{-1} are non-zero). But reversing the roles of σ_1 and σ_2 in the above argument, we have

$$\sigma_1^{-1}(i) \neq \sigma_2^{-1}(j)$$

for all $i \neq j$. Hence $\sigma_1 = \sigma_2$ and $P_1 = P_2$.

Hence, we have proven the Bruhat decomposition of $GL_m(\mathbb{C})$. Furthermore, we observe that:

(4.7) *Proposition.* $\dim_{\mathbb{C}}(U_w^-) = \ell(w)$, for $w \in \Sigma_m$.

Proof. Recall, that

$$\ell(w) = |\{(i < j): w(i) > w(j)\}|$$

It is easy to compute that the (i,j)th entry of wU^-w^{-1} is $t_{w(i),w(j)}$, where $(t_{i,j}) = U^-$. Hence, if $i < j$, then the (i,j)th entry is non-zero only if $w(i) > w(j)$, since elements of U^- are lower triangular. Hence, when we intersect with U, we find there are $\ell(w)$ coordinates that are non-trivial.

Now under the identification $G/B \approx \mathrm{Flag}(\mathbb{C}^m)$ the trivial coset x_0 goes to the standard flag S. To show that Bwx_0 and X_w correspond, it suffices to observe

1. If S denotes the standard flag, then $\sigma(wS) = w$.
2. If F is a flag, and $\beta \in B$, then $\sigma(\beta F) = \sigma(F)$.

We leave the verification of these facts as an easy exercise for the reader.

We have mentioned that the natural projection map $p_k : \mathrm{Flag}(\mathbb{C}^{n+k}) \to G_k(\mathbb{C}^{n+k})$ induces an injective map on cohomology. This is actually a special case of a more general Lie-theoretic phenomenon. We explain this now.

Recall from (I, §5) that W denotes the parabolic subgroup of a Coxeter group (W,S) generated by a subset θ of S. There is a parallel notion for semi-simple Lie groups G. A subgroup P of G is called a *parabolic* if P contains a conjugate of a Borel subgroup. The following result is basic.

(4.8) *Theorem.* ([14, p. 29]). Every parabolic in G is conjugate to some $P_\theta = BW_\theta B$, $\theta \subset S$.

These P_θ's are often called *standard* parabolics; clearly $P_\phi = B$, $P_S = G$.

Example. Let $G = GL_m$, so that $W = \Sigma_m$, $S = \{s_1,\ldots,s_{m-1}\}$ as in (I, 1.3a). If $\theta = \{s_{i_1},\ldots,s_{i_k}\} \subset S$, then

$$P_\theta = \begin{pmatrix} GL_{i_1} & & * & * \\ & GL_{i_2-i_1} & & \\ & & \ddots & * \\ \bigcirc & & & GL_{m-i_k} \end{pmatrix}$$

In particular, if $\theta = \{s_k\}$ one obtains a maximal parabolic

$$P_k = \begin{pmatrix} GL_k & * \\ \bigcirc & GL_{n-k} \end{pmatrix}$$

Since the isotropy group of a k-plane is P_k we can identify the Grassmannian as a homogeneous space

$$GL_{n+k}/P_k \xrightarrow{\approx} G_k(\mathbb{C}^{n+k})$$

We now record

(4.9) *Theorem.* The natural projection $\pi_\theta : G/B \to G/P_\theta$ induces a surjection

$$(\pi_\theta)_* : H_*(G/B) \twoheadrightarrow H_*(G/P_\theta)$$

and an injection

$$\pi_\theta^* : H^*(G/P_\theta) \rightarrowtail H^*(G/B)$$

Proof. See [6 , Thm. 5.5].

Geometrically, it is not difficult to see what is going on. The B-orbits in G/P_θ are obtained by collapsing together B-orbits BwB/B in G/B if their w's lie in the same left W_θ- coset. These new B-orbits are a cell-decomposition for G/P_θ. The cells are now indexed by the minimal length left coset representatives of W_θ in W; namely W^θ (cf. I, §5). We return to this idea in IV, §4.

§5. COMPLEMENTS

We briefly describe (another) four ways of thinking about the cohomology of $G_k(\mathbb{C}^{n+k})$. Three of these approaches involve different points of view on cohomology altogether; De Rham cohomology (5.1), Lie algebra cohomology (5.2) and the Chow ring (5.3). The fourth ties up $H^*(G_k(\mathbb{C}^{n+k}))$ with a classical treatment of symmetric functions due to Jacobi (5.4). We make no claim of doing justice to any of these subjects and heartily encourage the interested reader to pursue the suggested references for the complete story. We offer propaganda for (5.3) and return to it in Chapter V.

(5.1) Suppose X is a complex manifold. Let $\{\mathcal{a}^p(X), d^p\}$ be the de Rham complex of holomorphic differential p-forms on X; d^p the exterior derivative of forms. There is a de Rham map

$$D: \mathcal{a}^*(X) \to C^*(X)$$

where $C^*(X)$ denotes the usual cochain complex on X. It is given by

$$D(\omega)(\sigma) = \int_\sigma \omega$$

where $\omega \in \mathcal{a}^p(X)$ and $\sigma \in C_p(X)$. (The fact that D is a chain map is a generalization of the fundamental theorem of calculus). De Rham's theorem asserts that D induces an isomorphism on homology, in particular a cohomology class can be thought of as a differential form. The cup product of cohomology classes corresponds to the wedge product of differential forms. Stoll's monograph [150] systematically develops the cohomology of $G_k(\mathbb{C}^{n+k})$ from this point of view. The Schubert classes are identified with explicit differential forms. Stoll exploits a notion of fiber integration on singular varieties to prove the Pieri formula (3.6) (see [150, Ch. 7]). An interesting application is then developed concerning the "Schubert zeros" of sections of holomorphic vector bundles (cf. [18]).

(5.2) Kostant [88] describes the cohomology of $G_k(\mathbb{C}^{n+k})$ (and a large class of other homogeneous spaces G/H) in terms of certain Lie algebra (co-)homology groups. We restrict ourselves here to the case of the flag manifold G/B. The idea is roughly analogous to (5.1) in that the Schubert classes are pinned down by certain differential forms. But, Kostant identifies

these forms in terms of the homology of the nilradical $n = \mathit{Lie}(U)$. ($\mathit{Lie}(G)$ denotes the Lie algebra of G).

Firstly, by the van Est theorem, one can identify $H^*(G/B)$ with the relative Lie algebra cohomology group $H^*(g, b)$. The corresponding cochain complex can be thought of as the K-invariant complex differential forms on G/B. (K is a maximal compact form of G that provides a $*$-operation on g). It is shown [88 , Thm. 4.5] that there is an isomorphism

$$(*) \qquad (H_*(n) \otimes H_*(n^*)) = H^*(g,b)$$

where h is the Cartan subalgebra b/n. In an earlier paper, Kostant computed the irreducible components of $H_*(n)^h$ and they are indexed by $w \in W$. These components determine a basis $\{s_w\}$ for the left-hand side of (*). Kostant also picks out a "harmonic" representative for each s_w with respect to a suitable Laplacian. Finally, it is shown [88 , Thm. 6.15] that s_w is precisely a scalar multiple of the Schubert class X_w. An integral formula is given for this scalar, though an explicit formula is available now [86 , p. 357]; e.g. for $w = s_\alpha$, it is simply $2(\alpha,\alpha)^{-1}$. Koch [86] has further studied the multiplication of Schubert cycles from this point of view.

(5.3) It was pointed out in §3 that cup-product in cohomology is dual to intersection product in homology. If X is a non-singular projective variety over an algebraically closed field k (not necessarily of characteristic zero) one can form the *Chow* (or *intersection*) *ring* $A^*(X)$ [29]. The group $A^p(X)$ consists of formal linear combinations of irreducible codimension p subvarieties of X modulo rational equivalence (cf. [62 ,p. 426]). In particular, $A^1(X)$ is the Picard group of divisor classes. Intersecting subvarieties (after they have been moved into general position) induces a product

$$A^p(X) \otimes A^q(X) \to A^{p+q}(X)$$

This is the ring structure on $A^*(X)$. Now suppose $X = G/B$. The Bruhat decomposition still goes through over k. Let X_w denote the subvariety $\overline{B_{w_0w}B/B_w}$ (this is just Poincaré duality, cf. (IV, 2.9)). Since the dimension of X_{w_0w} is $\ell(w_0w) = \ell(w_0) - \ell(w)$, $X_w \in A^{\ell(w)}(G/B)$. We again get that the X_w are a $\mathbb{Z}$-basis for the Chow ring of G/B. We will adopt this point of view in Chapter V. Formally, this is identical to the Schubert picture.

(5.4) There is a remarkable relationship between the cohomology ring of a Grassmannian and the combinatorics of symmetric functions. Suppose we consider a polynomial ring $\mathbb{Z}[X_1,\ldots,X_n]$ in finitely many variables. There are several alternative generating sets for the subalgebra invariant under the action of the symmetric group Σ_n permuting the X_i's. There are the familiar elementary symmetric functions:

$$s_j(X_1,\ldots,X_n) = \sum X_{i_1}\ldots X_{i_j}$$

where the summation varies over $1 \le i_1 < \cdots < i_j \le n$, $1 \le j \le n$. They have a generating function

$$\prod_{i=1}^{n} (1 + X_iT)$$

If the generating function is replaced by

$$\prod_{i=1}^{n} (1 - X_iT)^{-1}$$

the coefficients of the T^j, $1 \le j \le n$, give another generating set

$h_1,\ldots,h_n$, the *complete symmetric functions.*

Now suppose $\alpha = (\alpha_1 > \cdots > \alpha_n)$ is a partition of $|\alpha| = \sum_{i=1}^{n} \alpha_i$ into n *distinct* parts. Then $\alpha = \lambda + \rho$ (pointwise) where $\rho = (n-1,n-2,\ldots,0)$ and $\lambda_1 = (\alpha_1 - (n-1), \alpha_2 - (n-2),\ldots,\alpha_n)$. We can antisymmetrize the monomial $X^\alpha = X_1^{\alpha_1}\ldots X_n^{\alpha_n}$, i.e

$$A_\alpha = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\sigma(X^\alpha)$$

The polynomial A_α is anti-symmetric, i.e.

$$\sigma(A_\alpha) = \operatorname{sgn}(\sigma)A_\alpha \qquad \sigma \in \Sigma_n$$

Now it is easy to see that the polynomial

$$S_\lambda = A_\alpha A_\rho^{-1} \qquad \lambda = \alpha + \rho$$

is a symmetric polynomial, being the quotient of two anti-symmetric ones. It is called an *S-function* (or *Schur function*)(see [101]). The S-functions $S_\lambda(X_1,\ldots,X_n)$, λ as above, form a $\mathbb{Z}$-basis for the algebra of symmetric functions. In any case, it should be possible to write each polynomial S_λ as a polynomial in the complete symmetric functions. Jacobi [79] found such an expression; it is often referred to as the *Jacobi-Trudi identity*.

$$S_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n}$$

The reader should not overlook the striking similarity between this result and the Giambelli formula (3.5). Here the S_λ corresponds to the Schubert classes and the h_i to the special Schubert classes. Is there an analogue of the Pieri formula? Indeed, there is an even more general result called the *Littlewood-Richardson rule*. It allows one to compute the coefficients $C^\eta_{\lambda\mu}$ in the expansion

$$S_\lambda \cdot S_\mu = \sum_\eta c^\eta_{\lambda\mu} S_\eta$$

A (complete) proof can be found in either Macdonald's book [101, p. 68] or Schützenberger [126]. When $S_\mu = h_i$ (letting $\mu = (i)$) one obtains a precise formal analogue of the Pieri formula (3.6). According to Stanley [137, p. 238] this connection was first observed by Lesieur [46] and rediscovered many times since. Explanations for this coincidence have been suggested by Horrocks [73] and Carrell and Lieberman [23]. Work of Lascoux [92],[93] has exploited the combinatorics of Schur functions to understand the geometry of Schubert varieties. We recommend [137] for a more detailed exposition of this circle of ideas and further references.

IV Schubert calculus of the coinvariant algebra

We now have all the tools at our disposal for the grand synthesis. Suppose W is a finite Coxeter group. The Borel picture identifies the coinvariant algebra S_W with the cohomology of a flag manifold G/B. On the other hand, the Schubert picture provides a Schubert calculus for the cohomology of G/B. In this chapter, we avoid the geometric intermediary and produce the Schubert calculus directly for S_W. One advantage is an extension to the non-crystallographic Coxeter groups. Another is that once one has Borel's theorem the Schubert calculus becomes an immediate algebraic corollary without any dependence on the Bruhat decomposition. We work directly with the length function and Bruhat order on the Coxeter groups (I, §§1 and 6) and the (anti-) invariant theory for these groups described in (II, §§3 and 4). We avoid any mention of Lie groups, homogeneous spaces, etc. Notice although the definition of the coinvariant algebra S_W depends only on W, the Schubert calculus depends on the choice of a geometric realization (Δ,Σ) (see I, §3). For example, the coinvariant algebra of the hyperoctahedral group supports two different Schubert calculi coming from the root systems of type B_n and C_n.

In section 1 we begin the program of finding an algebraic substitute for the Schubert varieties. Intuitively this is obtained by viewing homology classes as linear functionals on $S(V)$. By dualizing, we get algebraic models for the Schubert cohomology classes and show they form a basis for S_W. This result gives another expression for the Poincaré series of the coinvariant algebra.