Sections 2 and 3 are devoted to proving appropriate versions in S_W of the Giambelli and Pieri formulas, respectively, using the machinery developed in §1. In addition, a version of Poincaré duality is derived coming from the involution $w \to w_0 w$ on the Coxeter group W. We also exhibit a concrete computation.

If $\theta \subset S$, the parabolic subgroup W_θ acts on S_W and we compute its action and its invariants in section 4. This leads to a relative basis theorem for $S_W^{W_\theta}$.

In section 5 we apply the machinery of the preceding sections to analyze in detail the $\Sigma_k \times \Sigma_n$-invariants in the coinvariant algebra of Σ_{n+k}. By restricting the Pieri formula for $S_{\Sigma_{n+k}}$ down to this subalgebra we obtain an alternative algebraic derivation of the classical Pieri formula for the Grassmannian (III, 3.6).

Finally, in section 6 we collect two loose ends. First, we give a heuristic account of the result of Bernstein, Gelfand and Gelfand [6] that insures that the algebraic construction of §1 agrees with the geometric Bruhat decomposition. Second, we compute the torsion primes of G following Demazure [38].

§1. BASIS THEOREM

Our first goal is to produce an algebraic substitute for the cohomology algegra and the Schubert classes. The ideas required to do this seem to have been independently (and almost simultaneously) discovered by Demazure [38], [39] and Bernstein, Gelfand and Gelfand [6]. Most of this section is lifted directly from [38].

We fix a geometric realization (Δ,Σ) of (W,S) so that we can speak of positive roots, weights, etc. (see I, §3).

We begin with a small amount of motivation. If X_w is a Schubert class in $H_{2\ell(w)}(G/B)$ (see III, §4) we can define a corresponding linear functional D_w on $S(V)$. This functional vanishes on $S_j(V)$, $j \neq \ell(w)$, and on $S_{\ell(w)}(V)$, $D_w(f) = \langle X_w, c(f)\rangle \in \mathbb{C}$, where $\langle\ ,\ \rangle$ is the usual Kronecker pairing of homology and cohomology. We will construct algebraically the linear functionals that arise from the Schubert classes in this fashion. That they actually do arise in this way is a geometric theorem of Bernstein, Gelfand and Gelfand [6] which we discuss in §6. In any case what we do construct is *some* Schubert type description of the coinvariant algebra S_W.

The functionals we need have already been introduced in the course of our proof of the Chevalley theorem (II, §3). Recall that, if $\alpha \in V$, we can define (analogous to (II, 3.4))

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}$$

where $f \in S(V)$. This operator on $S(V)$ clearly reduces the grading by 1. If $\varepsilon : S(V) \to \mathbb{C}$ denotes evaluation at 0, then $\varepsilon\Delta_\alpha$ is the correct model for the Schubert class X_{s_α}, $\alpha \in \Sigma$. We will exploit these Δ-operators to algebraically reconstruct the map $c : S(V) \to H^*(G/B)$ (except that our map preserves degree). We begin with an omnibus lemma.

(1.1) *Lemma*. If $\alpha \in V$, $\omega \in S_1(V)$, $u,v \in S(V)$, $\varphi \in \mathrm{Aut}\, V$, then

(a) $s_\alpha \Delta_\alpha = \Delta_\alpha$

(b) $\Delta_\alpha^2 = 0$

(c) $\mathrm{Ker}(\Delta_\alpha) = S(V)^{\langle s_\alpha\rangle}$ (i.e. s_α-invariants)

(d) $\varphi\Delta_\alpha\varphi^{-1} = \Delta_{\varphi(\alpha)}$

(e) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$

(f) $\Delta_\alpha(I_W) \subseteq I_W$

(g) $\Delta_\alpha(\omega) = (\omega, \alpha^\vee)$

(h) $[\Delta_\alpha, \omega^*] = (\omega, \alpha^\vee) s_\alpha$

Proof. (a)-(d) are straightforward and left as an exercise for the reader. (e) is a restatement of (II, 3.4*). For (f), suppose $f \in S(V)$, $u \in S(V)^W_+$. By (c) and (e), $\Delta_\alpha(fu) = \Delta_\alpha(f)u \in I_W$. For (g) we have

$$\Delta_\alpha(\omega) = \frac{\omega - s_\alpha(\omega)}{\alpha} = \frac{\omega - (\omega, \alpha^\vee)\alpha)}{\alpha} = (\omega, \alpha^\vee)$$

Finally for (h), if $u \in S(V)$, then

$$\begin{aligned}[\Delta_\alpha, \omega^*](u) &= \Delta_\alpha \omega^*(u) - \omega^* \Delta_\alpha(u) \\ &= \frac{\omega u - s_\alpha(\omega u)}{\alpha} - \omega\left(\frac{u - s_\alpha(u)}{\alpha}\right) \\ &= \Delta_\alpha(\omega) s_\alpha(u)\end{aligned}$$

so (g) completes the argument.

The immediate goal is to show that it is legitimate to define $\Delta_w = \Delta_{\alpha_1} \cdots \Delta_{\alpha_k}$ where $w = s_{\alpha_1} \ldots s_{\alpha_k}$ is a reduced decomposition of $w \in W$. The strategy is to identify Δ_{w_0}, where w_0 is the longest word (which yields dividends in §2) and then induct down using the rank 2 Coxeter groups. We let d denote $\prod_{\beta \in \Delta_+} \beta \in S_N(V)$. Recall (II, §4) this is precisely the generator of $S(V)^{-W}$ as a free, rank 1 $S(V)^W$-module!

(1.2) *Proposition.* Suppose $w_0 = s_{\alpha_1} \ldots s_{\alpha_N}$. Then

$$\Delta_{\alpha_1} \cdots \Delta_{\alpha_N} = d^{-1}((-1)^N w_0 + \sum_{w \neq w_0} q_w w)$$

where $q_w \in \overline{S(V)}$, the field of rational functions on V.

Proof. We have

$$\begin{aligned}\Delta_{\alpha_1} \ldots \Delta_{\alpha_N} &= \alpha_1^{-1}(1 - s_1) \ldots \alpha_N^{-1}(1 - s_N) \\ &= (-1)^N \alpha_1^{-1} s_1 \ldots \alpha_N^{-1} s_N + \sum_{w \neq w_0} a_w w\end{aligned}$$

where $s_i = s_{\alpha_i}$. It now remains to see what happens in the first term as we pass the reflections s_i over to the right. We get $(-1)^N(\prod_{i=1}^N s_1 \ldots s_{i-1}(\alpha_i))^{-1} s_1 \ldots s_N$. But by (I, 3.6) this is $(-1)^N d^{-1} w_0$ since $w_0^{-1} = w_0$ and $w_0 \Delta^+ = \Delta^-$. We now let $q_w = a_w d$ and we are done.

We now show that the vector space $S_N(V)$ decomposes into an invariant and an anti-invariant piece in order to identify the q_w's of (1.2).

(1.3) *Lemma.* If $u \in S_N(V)$, then $J(u) \equiv |W| u \pmod{I_W}$, where $J(u) = \Sigma(-1)^{\ell(w)} wu$.

Proof. For any $\alpha \in \Sigma$, $u\alpha \in I_W$ (II, 3.10). Writing $u\alpha = \Sigma u_i f_i$, $f_i \in S(V)^W_+$, we easily check $u + s_\alpha(u) = \Sigma \Delta_\alpha(u_i) f_i \in I_W$. Hence $s_\alpha(u) \equiv -u \pmod{I_W}$, so that $w(u) \equiv (-1)^{\ell(w)} u \pmod{I_W}$. The result follows from computing the appropriate sum over W.

(1.4) *Corollary.* $S_N(V) = (I_W)_N + \mathbb{C} \cdot d$.

Proof. Write $u = \frac{1}{|W|}(|W|u - J(u)) + \frac{1}{|W|} J(u)$ and observe $\frac{1}{|W|} J(u)$ is divisible by d, being in $S(V)^{-W}$.

We can now show

(1.5) *Proposition.* If φ is an $S(V)^W$-endomorphism of $S(V)$ that reduces the grading by N, then $d\varphi = \lambda J$, for some $\lambda \in \mathbb{C}$.

Proof. By (1.4), we can write $u \in S_N(V)$ as $\Sigma u_i f_i + \mu d$, with $f_i \in S(V)^W_+$, $u_i \in S(V)$, $\deg(u_i) < N$, $\mu \in \mathbb{C}$. Hence

$$\varphi(u) = \Sigma\varphi(u_i)f_i + \mu\varphi(d) = \mu\varphi(d) \quad ,$$

Similarly, $J(u) = \mu J(d)$, so $d\varphi(u) = \frac{\varphi(d)}{|W|}J(u)$, so we let $\lambda = \frac{\varphi(d)}{|W|}$.

Finally, we can show

(1.6) *Proposition.* If $w_0 = s_{\alpha_1}\dots s_{\alpha_N}$ is a reduced decomposition of the longest word then $\Delta_{\alpha_1}\dots\Delta_{\alpha_N} = d^{-1}J$.

Proof. By (1.5), $d\Delta_{\alpha_1}\dots\Delta_{\alpha_N} = \lambda J = \lambda(-1)^N w_0 + \sum_{w\neq w_0}(-1)^{\ell(w)}\lambda w$. Also by (1.2), $d\Delta_{\alpha_1}\dots\Delta_{\alpha_N} = (-1)^N w_0 + \sum_{w\neq w_0} q_w w$. By Dedekind's theorem, the w's are linearly independent as automorphisms of $\overline{S(V)}$, so $\lambda = 1$ and the result follows.

(1.7) *Proposition.* The endomorphisms $\Delta_w, w \in W$, are well-defined.

Proof. It suffices to show that for all $\alpha,\beta \in \Sigma$

$$\Delta_\alpha\Delta_\beta\Delta_\alpha \dots = \Delta_\beta\Delta_\alpha\Delta_\beta \dots$$

with $m_{\alpha\beta}$ terms on each side. But the rank 2 root systems have $s_\alpha s_\beta s_\alpha \dots = s_\beta s_\alpha s_\beta \dots$ as their longest word, so (1.6) completes the argument.

Now we let $\mathbb{\Delta}$ denote the algebra of endomorphisms of $S(V)$ generated by the Δ_α, $\alpha \in \Sigma$, and multiplication operators ω^*, $\omega \in S_1(V) = V^*$. Clearly, $\mathbb{\Delta}_W$ is an $S(V)$-module. We let $\bar{\mathbb{\Delta}}_W$ denote the subalgebra of $S(V)^*$ obtained by applying ε to every operator in $\mathbb{\Delta}_W$. The composition

$$S(V) \to S(V)^{**} \xrightarrow{i^*} \bar{\mathbb{\Delta}}_W^*$$

is our model of the map c and, following Demazure, we christen $H_W = \bar{\mathbb{\Delta}}_W^*$,

the *cohomology of the root system* (Δ, Σ). We claim

(1.8) *Proposition.* $\mathbb{\Delta}_W$ is free as an $S(V)$-module, with basis $\{\Delta_w : w \in W\}$.

Proof. To show the Δ_w generate $\mathbb{\Delta}_W$, it suffices to show that in the composition $\Delta_\alpha\cdot\omega^*$, Δ_α can be moved to the right. But by (1.1e)

$$\Delta_\alpha\omega^*(v) = \Delta_\alpha(\omega v) = \Delta_\alpha(\omega)v + s_\alpha(\omega)\Delta_\alpha(v)$$

so $\Delta_\alpha\omega^* = \Delta_\alpha(\omega)^* + s_\alpha(\omega)^*\Delta_\alpha$. The Δ_w's are linearly independent operators on $\overline{S(V)}$, hence also on $S(V)$.

Before we prove the main result we need the following fact.

(1.9) *Lemma.* If I is a graded ideal of $S(V)$ containing I_W and containing no multiple of d then $I = I_W$.

Proof. By (II, 3.10) $(I_W)_n = S_n(V)$ for all $n > N$. Hence $I_n = (I_W)_n$ for all $n > N$. If $n = N$ (1.4) completes the argument. But if $u \in I'_n$, $n < N$, say $n = N-1$, $u\alpha \in I'_N = (I_W)_N$, for all $\alpha \in \Sigma$ and the proof of (1.3) implies $u \in (I_W)_n$. We are finished by induction.

(1.10) *Basis Theorem.* The algebra H_W possesses a basis $\{X_w : w \in W\}$ dual to $\{\varepsilon\circ\Delta_w : w \in W\}$. The map c is described by

(1.10*) $$c(u) = \sum_{w\in W}\varepsilon\Delta_w(u)X_w$$

Furthermore, it is onto with kernel $= I_W$, so induces an isomorphism $S_W \approx H_W$.

Proof. The only assertion that is not obvious concerns the kernel. But by (1.9) it suffices to check $c(d) \neq 0$. We can compute, by (1.6)

$$\Delta_{w_0}(d) = d^{-1}J(d) = d^{-1}|W|d = |W| \neq 0$$

so by (1.10*) the proof is complete.

We can now justify the remark following (3.10) of Chapter II and derive an interesting identity. Namely:

(1.11) *Corollary.* The Poincaré series of the coinvariant algebra S_W is

(a) $$PS(S_W,t) = \sum_{w\in W} t^{\ell(w)}$$

and hence

(b) $$\sum_{w\in W} t^{\ell(w)} = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$$

where $d_1,\ldots,d_n$ are the fundamental degrees of W.

Proof. The first assertion is a consequence of (1.10) and the second follows from (II, 3.10).

Remark. The left hand side of (1.11b) is sometimes called the *Poincaré series* of the Coxeter group W. It is always a rational function of t (cf. [149], [19, p. 45]) and often satisfies a functional equation. Computing such generating functions often yields interesting identities. Macdonald [99] has written down the Poincare series of the affine Weyl groups in terms of a height function on the root system. There is also recent work of J. Cannon on the hyperbolic Coxeter groups [22].

§2. GIAMBELLI FORMULA

In Chapter III, we saw that among the Schubert classes, there existed special

Schubert classes that algebraically generate the cohomology. We would like analogous classes in the algebra H_W. Indeed, any basis $\beta_1,\ldots,\beta_n$ of the vector space $V \approx V^*$ yields 1-dimensional algebraic generators for $S(V)$ and hence algebraic generators for H_W. The most natural choice however turns out to be the basis $\{\omega_\alpha\}_{\alpha\in\Sigma}$ of fundamental weights (see I, 3.9). This is made clear by (ii) of the following.

(2.1) *Lemma.*

(i) $\Delta_\beta(\omega_\alpha) = \delta_{\alpha\beta}$

(ii) $c(\omega_\alpha) = X_{s_\alpha}$

(iii) $c(\beta) = \sum_{\alpha\in\Sigma} (\beta,\alpha^\vee)X_{s_\alpha}$

Proof. (i) follows from (1.1c) and the definition of the ω_α's. For (ii), we compute using (i):

$$c(\omega_\alpha) = \sum_{w\in W} \varepsilon\Delta_w(\omega_\alpha)X_w$$

$$= \sum_{\beta\in\Sigma} \Delta_\beta(\omega_\alpha)X_w = X_{s_\alpha}$$

Finally, (iii) follows from the expansion $\alpha = \sum_{\beta\in\Sigma} (\alpha,\beta^\vee)\omega_\beta$ and (ii).

Remark. If we identify $(H_W)_1$ with $\mathrm{Pic}(G/B)$, the formula (2.1 (iii)) appears in Iversen's work on algebraic groups [77].

Hence, the goal is to find for every $w \in W$ a polynomial $Q_w(X_1,\ldots,X_n)$ such that $c(Q_w(\omega_{\alpha_1},\ldots,\omega_{\alpha_n})) = Q_w(X_{s_{\alpha_1}},\ldots,X_{s_{\alpha_n}}) = X_w$. (Of course, such a Q_w is not uniquely determined). Certainly, it will suffice to find a polynomial $P_w(X_1,\ldots,X_n)$ satisfying $c(P_w(\alpha_1,\ldots,\alpha_n)) = X_w$. Then Q_w will be determined by (2.1 (iii)) and the "Cartan matrix" $(\alpha,\beta^\vee)_{\alpha,\beta\in\Sigma}$. We follow this strategy here. First, we will give an explicit form for Q_{w_0} and then

show that all other Q_w can be obtained by applying appropriate Δ-operators. We begin with

(2.2) *Lemma.* Δ is quasi-multiplicative, i.e.

$$\Delta_w \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w') \\ 0 & \text{otherwise} \end{cases}$$

Proof. The first clause is immediate since the condition means that reduced decompositions of w and w' can be juxtaposed to yield a reduced decomposition of ww'. To show the second part we induct on $\ell(w)$. If $\ell(w) = 1$, so that $w = s_\alpha$, for some $\alpha \in \Sigma$, then by (I,1.6) $\ell(s_\alpha w') = \ell(w') - 1$. Since $w' = s_\alpha(s_\alpha w')$ and $\ell(w') = 1 + (\ell(w') - 1) = \ell(s_\alpha) + \ell(s_\alpha w')$ by the first part we get $\Delta_{w'} = \Delta_{s_\alpha}\Delta_{s_\alpha w'}$. But by (1.1c)

$$0 = \Delta_{s_\alpha}\Delta_{s_\alpha}\Delta_{s_\alpha w'} = \Delta_{s_\alpha}\Delta_{w'}$$

If $\ell(w) > 0$, we write $w = s_\alpha v$ with $\ell(w) = 1 + \ell(v)$. We have two cases

1) $\ell(vw') < \ell(v) + \ell(w')$. Then, by induction,

$$\Delta_{s_\alpha v}\Delta_{w'} = \Delta_{s_\alpha}\Delta_v\Delta_{w'} = \Delta_{s_\alpha}\cdot 0 = 0$$

2) $\ell(vw') = \ell(v) + \ell(w')$. Then, we have

$$\ell(s_\alpha vw') < \ell(s_\alpha v) + \ell(w') = 1 + \ell(v) + \ell(w') = 1 + \ell(vw')$$

so, by (I, 1.6), $\ell(s_\alpha vw') < \ell(vw')$ and induction again yields $\Delta_{s_\alpha}\Delta_{vw'} = 0$; hence

$$\Delta_{s_\alpha v}\Delta_{w'} = \Delta_{s_\alpha}\Delta_v\Delta_{w'} = \Delta_{s_\alpha}\Delta_{vw'} = 0$$

This completes the proof.

(2.3) *Corollary.* $\Delta_{w'}\Delta_{w^{-1}w_0} = \delta_{w,w'}\Delta_{w_0}$ for all $w,w' \in W$.

Proof. If $w = w'$, the result follows immediately from (2.2). For dimension reasons, it suffices to consider $w' \neq w$ with $\ell(w') = \ell(w)$. But we can compute by (I, 1.3)

$$\begin{aligned} \ell(w'w^{-1}w_0) &= \ell(w_0) - \ell(w'w^{-1}) < \ell(w_0) \\ &= \ell(w') + (\ell(w_0) - \ell(w)) \\ &= \ell(w') + \ell(w^{-1}w_0) \end{aligned}$$

Hence, again by (2.2), $\Delta_{w'}\Delta_{w^{-1}w_0} = 0$.

We can now dualize this result to the following assertion. From it one can read off the desired polynomials Q_w.

(2.4) *Theorem.* (Giambelli formula). In the algebra H_W, for all $w \in W$

$$c(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|})) = X_w$$

Hence, in particular, $c(\frac{d}{|W|}) = X_{w_0}$.

Proof.

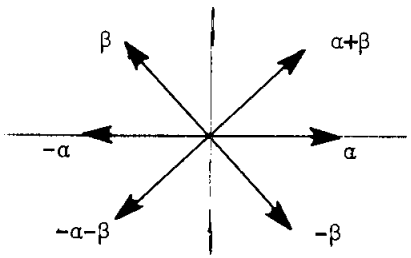
$$\begin{aligned} c(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|})) &= \sum_{w' \in W} \varepsilon\Delta_{w'}(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|})) \\ &= \sum_{\substack{w' \in W \\ \ell(w') = \ell(w)}} \delta_{w,w'}\varepsilon\Delta_{w_0}(\tfrac{d}{|W|})X_{w'} \\ &= \tfrac{1}{d}J(\tfrac{d}{|W|})X_w = X_w \end{aligned}$$

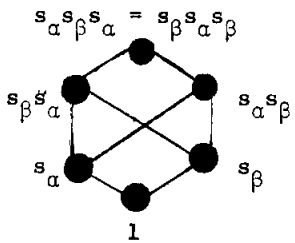
One dividend of our work on relative invariants (I, §4) is the following alternative expression for X_{w_0}.

(2.5) *Proposition.* Let $f_1,\ldots,f_n$ be the fundamental invariants of W. Then if $D = \det(\frac{\partial f_i}{\partial e_j})$ is the Jacobian of these polynomials, there is a $\lambda \in \mathbb{C}$ such that $c(\lambda D) = X_{w_0}$.

Proof. This follows from (II,4.4), since d is exactly the generating anti-invariant.

(2.5.1) *Example.* Let $W = W(A_2) = \Sigma_3$ where A_2 is the root system in $\{\sum_{i=1}^{3} \lambda_i e_i : \sum_{i=1}^{3} \lambda_i = 0\} \subseteq \mathbb{R}^3$ with simple roots $\Sigma = \{\alpha = e_1 - e_2, \beta = e_2 - e_3\}$ and positive roots $\Delta^+ = \{\alpha, \beta, \alpha+\beta\}$





We have $X_{w_0} = c(\frac{1}{6}\alpha\beta(\alpha+\beta))$. As a check, we compute the Jacobian of the fundamental invariants. (In general this Jacobian is the Vandermonde determinant of the e_i's). Recall $\sigma_2 = -(e_2+e_3)(e_2+e_3) + e_2e_3$ and $\sigma_3 = -(e_2+e_3)e_2e_3$, (where we eliminated $e_1 = -(e_2+e_3)$ and hence also σ_1). Then

$$D = 3(e_2^2e_3 - e_3^2e_2) + 2(e_2^3 - e_3^3) = d$$

as can easily be checked. We can now apply the Δ-operators to get:

$$\Delta_\alpha(\tfrac{d}{6}) = \tfrac{1}{3}\beta(\alpha+\beta)$$

and

$$\Delta_\beta\Delta_\alpha(\tfrac{d}{6}) = \tfrac{1}{3}\Delta_\beta(\beta(\alpha+\beta)) = \tfrac{1}{3}(2\alpha+\beta)$$

so that $X_{s_\alpha s_\beta} = c(\frac{1}{3}\beta(\alpha+\beta))$ and $X_{s_\alpha} = c(\frac{1}{3}(2\alpha+\beta)) = \omega_\alpha$.

Since the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ we have

$$\alpha = 2\omega_\alpha - \omega_\beta$$
$$\beta = -\omega_\alpha + 2\omega_\beta$$

so, for example

$$\begin{aligned} X_{s_\alpha s_\beta} &= \tfrac{1}{3}(-X_{s_\alpha} + 2X_{s_\beta})(X_{s_\alpha} + X_{s_\beta}) \\ &= \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta}X_{s_\alpha} + 2X_{s_\beta}^2) \end{aligned}$$

which will be confirmed in the next section in a different way with the Pieri formula.

Remark. It is possible to derive another expression for the fundamental class X_{w_0} by examining the degree N homogeneous part of Weyl's denominator formula (see [6, p. 17]). It is

$$X_{w_0} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where ρ is the (familiar) half-sum of the positive roots (equivalently, $\rho = \sum_{\alpha\in\Sigma} \omega_\alpha$).

As an application of the machinery of this section we compute a version of Poincaré duality for the coinvariant algebra S_W. We begin with the following easy observations.

(2.6) *Lemma.* If $\alpha \in \Sigma$ and $u,v \in S(V)$, then

$$\Delta_\alpha(\Delta_\alpha(u)v) = \Delta_\alpha(u\Delta_\alpha(v))$$

Proof. Invoke (1.1 a,b and e).

(2.7) *Lemma.* If $w \in W$ and $u,v \in S(V)$ then

$$\Delta_{w_0}(\Delta_w(u)v) = \Delta_{w_0}(u\Delta_{w^{-1}}(v)).$$

Proof. By induction on $\ell(w)$ it suffices to check it for $w = s_\alpha$, $\alpha \in \Sigma$. But, by (2.2) and (2.6),

$$\begin{aligned}\Delta_{w_0}(\Delta_w(u)v) &= \Delta_{w_0w}\Delta_w(\Delta_w(u)v)\\ &= \Delta_{w_0w}\Delta_w(u\Delta_w(v))\\ &= \Delta_{w_0}(u\Delta_w(v))\end{aligned}$$

so the proof is complete.

(2.8) *Proposition.* If $w \in W$ and $v \in S(V)$ is homogeneous of degree $N - \ell(w)$ then, in S_W

$$X_w \cdot c(v) = \Delta_{w_0w}(v)X_{w_0}$$

Proof. We compute using (2.4), (1.10*) and (2.7).

$$\begin{aligned}X_w \cdot c(v) &= c(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|}))\cdot c(v)\\ &= c(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|})\, v)\\ &= \Delta_{w_0}(\Delta_{w^{-1}w_0}(\tfrac{d}{|W|})v)\\ &= \Delta_{w_0}(\tfrac{d}{|W|}\Delta_{w_0w}(v))\\ &= \Delta_{w_0w}(v)X_{w_0}\end{aligned}$$

It is now easy to show that X_{w_0w} is the "Poincaré dual" of X_w.

(2.9) *Theorem.* If $w,w' \in W$ then in the algebra S_W

$$X_w \cdot X_{w_0w'} = \delta_{w,w'}X_{w_0}$$

Proof. Letting $v = \Delta_{w'^{-1}}(\frac{d}{|W|})$ in (2.8), we get

$$\begin{aligned}X_w \cdot X_{w_0w'} &= X_w c(\Delta_{w'^{-1}}(\tfrac{d}{|W|}))\\ &= \Delta_{w_0w}\Delta_{w'^{-1}}(\tfrac{d}{|W|})X_{w_0}\\ &= \delta_{w,w'}X_{w_0}\end{aligned}$$

by (2.3).

This result gives a precise description of Poincaré duality for the cohomology of the flag manifold G/B.

§3. PIERI FORMULA

Recall that the algebra of operators $\mathbb{A}_W$ was generated by the Δ_α's, $\alpha \in \Sigma$, and the multiplication operators ω^*, $\omega \in S_1(V)$. Using the $S(V)$-basis constructed in Section 1, if one composes such operators, say $\omega^*\Delta_w$ or $\Delta_w\omega^*$, it is possible to express them as $S(V)$-linear combinations of the Δ_g, $g \in W$. Of course, our eventual concern is with the algebra $\bar{\mathbb{A}}_W$ and $\varepsilon\omega^*\Delta_w = 0$. So if we compute the commutator $[\Delta_w,\omega^*]$, an application of ε will yield a

formula for $\varepsilon\Delta_w\omega^*$. Essentially, such a result is our Pieri formula for H_W disguised in its dual form.

In order for an induction argument to work smoothly it is advantageous to work with the slightly modified operator $w^{-1}\Delta_w$ (recall $W \subseteq \Delta_W$, since $s_\alpha = 1 - \alpha\Delta_\alpha$). The main result is

(3.1) *Theorem.* If $w \in W$, $\omega \in V^*$ then in End $S(V)$,

$$[w^{-1}\Delta_w,\omega^*] = \sum_{\substack{w' \to w \\ \gamma}} (w'^{-1}(\gamma)^\vee,\omega)w^{-1}\Delta_{w'}$$

We fix a reduced decomposition $w = s_1\cdots s_k$, where $s_i = s_{\alpha_i}$, and set $w_i = s_k\cdots s_i$, $1 \le i \le n$. We have the following easy observation.

(3.2) *Lemma.* Let $\theta_i = s_k\cdots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i)$, $1 \le i < k$, $\theta_k = \alpha_k$. Then

(i) $w^{-1}\Delta_w = \Delta_{\theta_1}\Delta_{\theta_2}\cdots\Delta_{\theta_k}$.

(ii) $s_{\theta_i}(w_{\hat{i}})^{-1} = w^{-1}$, where $w_{\hat{i}} = s_1\cdots\hat{s}_i\cdots s_k$.

Proof. By (1.1 d), we get

$$w^{-1}\Delta_w = s_k\cdots s_1\Delta_{\alpha_1}\cdots\Delta_{\alpha_k} = \Delta_{s_k\cdots s_2(\alpha_1)}s_k\cdots s_2\Delta_{\alpha_2}\cdots\Delta_{\alpha_k}$$

$$= \Delta_{\theta_1}s_k\cdots s_2\Delta_{\alpha_2}\cdots\Delta_{\alpha_k}$$

and induction completes the argument for (i). Finally, the second part follows precisely from (I, 3.6) applied to $w^{-1} = s_k\cdots s_1$.

Proof of 3.1. We compute

$$[w^{-1}\Delta_w,\omega^*] = [\Delta_{\theta_1}\cdots\Delta_{\theta_k},\omega^*]$$

$$= \sum_{j=1}^{k} \Delta_{\theta_1}\cdots\Delta_{\theta_{j-1}}[\Delta_{\theta_j},\omega^*]\Delta_{\theta_{j+1}}\cdots\Delta_{\theta_k}.$$

Let us call the jth summand P_j. Now, by (1.1h) we have $[\Delta_{\theta_j},\omega^*] = (\theta_j^\vee,\omega)s_{\theta_j}$. If we substitute this into P_j and drag the reflection s_{θ_j} over to the left we get

$$P_j = (\theta_j^\vee,\omega)\Delta_{\theta_1}\cdots\Delta_{\theta_{j-1}}s_{\theta_j}\Delta_{\theta_{j+1}}\cdots\Delta_{\theta_k}$$

$$= (\theta_j^\vee,\omega)s_{\theta_j}\Delta_{s_{\theta_j}(\theta_1)}\cdots\Delta_{s_{\theta_j}(\theta_{j-1})}\Delta_{\theta_{j+1}}\cdots\Delta_{\theta_k}$$

$$= (\theta_j^\vee,\omega)s_{\theta_j}(w_{\hat{j}})^{-1}\Delta_{w_{\hat{j}}}.$$

To see this final identity, we must argue for

$$\theta_{i,\hat{j}} = \begin{cases}\theta_i & i \ge j \\ s_{\theta_j}(\theta_i) & i < j\end{cases}$$

where $\theta_{i,\hat{j}} = s_k\cdots\hat{s}_j\cdots s_{i+1}(\alpha_i)$, i.e. the θ_i's for $w_{\hat{j}}$. The first assertion is easy and for $i < j$:

$$s_{\theta_j}(\theta_i) = s_k\cdots s_{j+1}s_js_{j+1}\cdots s_k(s_k\cdots s_{j+1}s_j\cdots s_{i+1}(\alpha_i))$$

$$= s_k\cdots\hat{s}_j\cdots s_{i+1}(\alpha_i) = \theta_{i,\hat{j}}.$$

Now, by (3.2 ii)

$$P_j = (\theta_j^\vee,\omega)w^{-1}\Delta_{w_{\hat{j}}}$$

and, also, $s_{w_{\hat{j}}(\theta_j)}w_{\hat{j}} = w_{\hat{j}}s_{\theta_j} = w$. In the notation of the Bruhat order (I, §6): $w_{\hat{j}} \xrightarrow{w_{\hat{j}}(\theta_j)} w$. Hence (I, 6.4, 6.6) allows us to reindex by the immediate subwords

$$\sum_{j=1}^{k} P_j = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)^\vee,w)w^{-1}\Delta_{w'}$$

where the computation $w'^{-1}(\gamma) = (w_j^{\wedge})^{-1}(w_j^{\wedge}(\theta_j)) = \theta_j$ verifies the coefficient. This completes the proof.

(3.3) *Corollary.* If $w \in W$, $\omega \in V^*$ then $\Delta_w \omega^* = w\omega^* w^{-1}\Delta_w + \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)^\vee, \omega)\Delta_{w'}$.

Proof. Multiply (3.1) by w.

(3.4) *Corollary.* $\varepsilon \circ \Delta_w \omega^* = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)^\vee, \omega)\varepsilon\Delta_{w'}$.

Proof. The first term on the right-hand side of (3.3) is annihilated by ε.

It is now easy to dualize and obtain

(3.5) *Theorem.* (Pieri formula). If $w \in W$, $\alpha \in \Sigma$, then in H_W

$$X_{s_\alpha} X_w = \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^\vee, \omega) X_{w'}$$

Proof. Choose u such that $\varepsilon\Delta_{w'}(u) = \delta_{ww'}$, (for example, the expression given by (2.4)). Then

$$\begin{aligned} X_{s_\alpha} X_w &= c(\omega_\alpha u) \\ &= \sum_{w' \in W} \varepsilon\Delta_{w'}(\omega_\alpha u) X_{w'} \\ &= \sum_{w' \in W} \varepsilon\Delta_{w'}\omega_\alpha^*(u) X_{w'} \\ &= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^\vee, \omega_\alpha)\varepsilon\Delta_g(u)\Big) X_{w'} \\ &= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^\vee, \omega_\alpha)\delta_{gw}\Big) X_{w'} \\ &= \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^\vee, \omega_\alpha) X_{w'} \end{aligned}$$

Now by (I, 6.1.c) we can rewrite (3.5) in the following equivalent form.

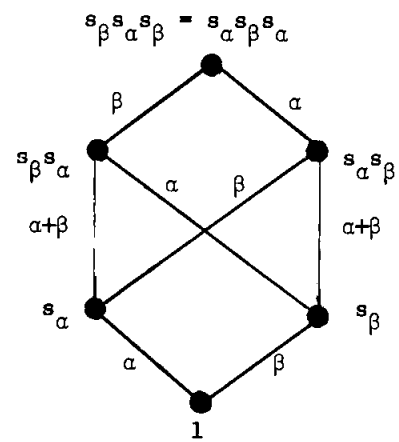
(3.6) *Corollary.* In H_W: $X_{s_\alpha} \cdot X_W = \sum_{\substack{\beta \in \Delta_+ \\ \ell(ws_\beta) = \ell(w)+1}} (\beta^\vee, \omega_\alpha) X_{ws_\beta}$

In practice, this expression is more convenient. We return to the situation examined in the last section.

Example. In H_{Σ_3}, we computed

(*) $$X_{s_\alpha s_\beta} = \frac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2)$$

We have the Bruhat order on Σ_3



where the notation $w \xrightarrow{\gamma} w'$ means $ws_\gamma = w'$. We can read off from (3.6) that

$$X_{s_\alpha}^2 = X_{s_\beta s_\alpha}$$
$$X_{s_\beta} X_{s_\alpha} = X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta}$$
$$X_{s_\beta}^2 = X_{s_\alpha s_\beta}$$

and this checks our earlier computation (2.5.1).

In Chapter V, §3 we will investigate the rich combinatorial structure inherent in formula (3.6).

§4. PARABOLIC INVARIANTS

If (W,S) is a Coxeter system and $\theta \subseteq S$, then (W_θ,θ) is also a Coxeter system and recall W_θ is called a *parabolic subgroup* of W. It is easy to see that a geometric realization (Δ,Σ) of (W,S) can be restricted to a geometric realization $(\Delta_\theta,\Sigma_\theta)$ of (W_θ,θ). Recall that in (I, §5) we showed that

$$\begin{aligned} W^\theta &= \{w \in W : \ell(ws) = \ell(w) + 1, \quad \text{for all} \quad s \in \theta\} \\ &= \{w \in W : \ell(ws_\alpha) = \ell(w) + 1, \quad \text{for all} \quad \alpha \in \Sigma_\theta\} \end{aligned}$$

forms a complete set of left coset representatives of W_θ in W and each has minimal length in its coset.

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in the coinvariant algebra H_W. According to our remark in Chapter III, for an appropriate choice of W_θ in Σ_{n+k}, we are really studying the cohomology of the Grassmann manifold. We return to this special situation in §5.

The most straightforward approach is to compute precisely the action of W on H_W. This is easily done by utilizing the computation (3.4).

(4.1) *Theorem.* The structure of H_W as a W-module is determined by

$$s_\alpha X_w = \begin{cases} X_w & \text{if } \ell(ws_\alpha) = \ell(w) + 1 \\ X_w - \sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^\vee, \alpha) X_{w'} & \text{if } \ell(ws_\alpha) = \ell(w) - 1 \end{cases}$$

where $w \in W$, $\alpha \in \Sigma$.

Proof. As in (4.5), choose u such that $\varepsilon\Delta_g(u) = \delta_{gw}$. Then since c is a W-map

$$\begin{aligned} s_\alpha X_w = c(s_\alpha u) &= \sum_{w' \in W} \varepsilon\Delta_{w'}(s_\alpha u) X_{w'} \\ &= \sum_{w' \in W} \varepsilon\Delta_{w'}(1 - \alpha^*\Delta_\alpha)(u) X_{w'} \\ &= X_w - \sum_{w'} (\varepsilon\Delta_{w'}, \alpha^*)\Delta_\alpha(u) X_{w'} \\ &= X_w - \sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^\vee, \alpha)\Delta_g\Delta_\alpha(u) X_{w'} \\ &= X_w - \sum_{\substack{g \xrightarrow{\gamma} w' \\ g(\alpha) \in \Delta^+ \\ gs_\alpha = w}} (g^{-1}(\gamma)^\vee, \alpha) X_{w'} \\ &= X_w - \sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^\vee, \alpha) X_{w'} \end{aligned}$$

Note, that the summation in the last line is non-vacuous if and only if $\ell(ws_\alpha) = \ell(w) - 1$. This completes the proof.

(4.2) *Corollary.* $X_w \in H_W^{W_\theta}$ if $w \in W^\theta$.

Proof. Immediate.

It remains to show that the X_w, $w \in W^\theta$, actually generate the W_θ-invariants. We use a dimension argument. Firstly

(4.3) *Lemma.* If V is the regular representation of a finite group G and H is a subgroup of G, then

$$\dim_{\mathbb{C}}(V^H) = |G|/|H|.$$

Proof. Let $\{e_g\}_{g \in G}$ be a basis for V, so that

$$g' \cdot e_g = e_{gg'}$$

Then, if $v = \sum_{g \in G} v_g e_g \in V^H$, we claim $v_g = v_{g'}$, if $g \equiv g' \pmod H$. Indeed, if $g = g'h$, $h \in H$, and if $k(v,e_g)$ denotes the coefficient of e_g in $v \in V$, then:

$$v_{g'} = k(v,e_{g'}) = k(h^{-1}v,e_{g'}) = v_g.$$

Hence, there are at most $|G|/|H|$ free parameters in determining $v \in V^H$ and clearly each choice gives an invariant. This finishes the argument.

(4.4) *Corollary.* (Basis theorem for $H_W^{W_\theta}$). $\dim_{\mathbb{C}} H_W^{W_\theta} = |W^\theta|$ and so the X_w, $w \in W^\theta$, are a $\mathbb{C}$-basis for $H_W^{W_\theta}$.

Proof. This follows from (4.2), (4.3) and (II, 3.14).

As in §1, we can use this basis result to derive an expression for the Poincaré series.

(4.5) *Corollary.* Suppose W has fundamental degrees $d_1,\ldots,d_n$ and W_θ has fundamental degrees $e_1,\ldots,e_m$. (Note W_θ is very often reducible). Then

(a) $$PS(S_W^{W_\theta},t) = \sum_{w \in W^\theta} t^{\ell(w)}$$

and

(b) $$\sum_{w \in W^\theta} t^{\ell(w)} = \frac{\prod_{i=1}^{n} (1-t^{d_i})}{(1-t)^{n-m} \prod_{j=1}^{m} (1-t^{e_j})}$$

Proof. (a) follows from (4.4) and (b) is a consequence of (1.11) and (I, 5.3).

Example. Suppose $W = \Sigma_{n+k}$ and $\theta = S - \{s_k\}$ so that $W_\theta = \Sigma_k \times \Sigma_n$ (see §5). The fundamental degrees of W are $2,\ldots,n$ and those of W_θ are $2,\ldots k$, $2,\ldots n$. Hence

$$PS(S_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n},t) = \frac{(1-t^{n+k})\cdots(1-t^{n+1})}{(1-t^k)\cdots(1-t)}$$

The polynomial on the right is the *Gaussian polynomial* $\left[{n+k \atop k}\right]$. Its coefficients are the Betti numbers of the complex Grassmann manifold (after replacing t by t^2).

Exercise. From (II, 3.13) check that the Poincaré series of W^θ, where W is the hyperoctahedral group and $\theta = \{s_1,\ldots,s_{n-1}\}$ (so that $W_\theta = \Sigma_n$) is given by

$$PS(W^\theta,t) = \begin{cases} \dfrac{(1-t^{n+2})(1-t^{n+4})\cdots(1-t^{2n})}{(1-t)(1-t^3)\cdots(1-t^{n-1})} & \text{if } n \text{ is even} \\[2ex] \dfrac{(1-t^{n+1})(1-t^{n+3})\cdots(1-t^{2n})}{(1-t)(1-t^3)\cdots(1-t^{n})} & \text{if } n \text{ is odd} \end{cases}$$

§5. GEOMETRY OF THE SYMMETRIC GROUP

In order to bring the abstract results of §§1-4 back down to earth we give a complete analysis of the coinvariant theory of the symmetric group Σ_{n+k} and its parabolic invariants. This has the pleasant consequence of giving a completely algebraic derivation of the classical Pieri formula discussed in Chapter III, §3.

We fix some notation. Let $W = \Sigma_{n+k}$, the Weyl group of type A_{n+k-1}. In Chapter I, §3 we wrote down the usual geometric realization of W. We adopt the notation from there. Furthermore, let s_{ij} be the reflection

corresponding to $e_i - e_j \in \Delta^+$ and $s_i = s_{i,i+1}$, $1 \le i < n+k$. Our first goal is to write down the Pieri formula for S_W. We begin with an easy length computation.

(5.1) *Lemma*. If $w \in W$, then

$$\ell(ws_{ij}) - \ell(w) = p_{ij}(2|I_{ij}| + 1)$$

where

$$p_{ij} = \begin{cases} +1 & \text{if } w(i) < w(j) \\ -1 & \text{if } w(j) < w(i) \end{cases}$$

$I_{ij} = \{i < z < j: w(z) \text{ is between } w(i) \text{ and } w(j)\}$. In particular, $\ell(ws_{ij}) = \ell(w) + 1$ if and only if (i) $w(i) < w(j)$ and (ii) there are no intermediate w-values, i.e. $I_{ij} = \phi$. (We abbreviate this pair of conditions by $w(i) \ll w(j)$).

Proof. Recall that the length function on Σ_{n+k} is given by $\ell(w) = \sum_{j=1}^{n+k-1} e_j(w)$, where $e_j(w) = |\{i > j: w(i) < w(j)\}|$, the number of inversions above j. Hence

$$\ell(ws_{ij}) - \ell(w) = (e_i' - e_i) + (e_j' - e_j) + \sum_{i<z<j} (e_z' - e_z)$$

where $e_\ell = e_\ell(w)$ and $e_\ell' = e_\ell(ws_{ij})$. Certainly right multiplication by s_{ij} does not affect the values of e_z below i or above j. Also

$$e_i' = e_j + |\{i \le z < j: w(z) < w(j)\}| = e_j + e$$
$$e_j' = e_i - |\{i < z \le j: w(z) < w(i)\}| = e_i - \bar{e}.$$

So we get

$$(e_i' - e_i) + (e_j' - e_j) = (e_j + e - e_i) + (e_i - \bar{e} - e_j)$$
$$= e - \bar{e} = p_{ij}(|I_{ij}| + 1)$$

It is also easy to see $e_z' - e_z = \begin{cases} p_{ij} & \text{if } z \in I_{ij} \\ 0 & \text{otherwise} \end{cases}$; putting all this together we get the result. The second assertion follows immediately.

We can now write down the Pieri formula (3.6) for S_W.

(5.2) *Proposition*. If $w \in W$, $1 \le i < n+k$, then in S_W

$$X_{s_i} \cdot X_w = \sum_{(b,t)} X_{ws_{bt}}$$

where (b,t) satisfies $b \le i < t$ and $w(b) \ll w(t)$.

Proof. By (3.6), $X_{ws_{bt}}$ appears with coefficient $((e_b - e_t)^\vee, \omega_i)$ if and only if $\ell(ws_{bt}) = \ell(w) + 1$. This is equivalent to $w(b) \ll w(t)$ by (5.1). Finally $(e_b - e_t)^\vee = \alpha_b^\vee + \cdots + \alpha_{t-1}^\vee$ so the first condition is also needed and the coefficient is correct.

Remark. The Poincaré dual of this formula appears in [108, p. 265].

We now identify the set of left coset representatives W^θ, where $\theta = S - \{s_k\}$. The corresponding subgroup W_θ is precisely $\Sigma_k \times \Sigma_n$. If $1 \le d_1 < \cdots < d_k \le n+k$ are k distinct numbers and $1 \le d_1' < \cdots < d_n' \le n+k$ is an ordered enumeration of their complement, then we define $(d_1, \ldots, d_k) \in \Sigma_{n+k}$ by

$$(d_1, \ldots, d_k)(i) = \begin{cases} d_i & 1 \le i \le k \\ d_{i-k}' & k+1 \le i \le k+n \end{cases}$$

(5.3) *Lemma.* $W_k^\theta = \{(d_1,\ldots,d_k): 1 \le d_1 < \cdots < d_k \le n+k\}$ and $\ell(d_1,\ldots,d_k) = \sum_{j=1}^{k} (d_j - j)$.

Proof. Clearly $\ell((d_1,\ldots,d_k)s_i) = \ell(d_1,\ldots,d_k) + 1$ for all $i \neq k$, by (5.1). Since $|W^\theta| = |W|/|W_\theta| = \binom{n+k}{k}$, the first assertion follows. For the second, we need only observe

$$e_j(d_1,\ldots,d_k) = \begin{cases} d_j - j & \text{if } j \le k \\ 0 & \text{otherwise} \end{cases}$$

According to the ideas of Chapter III, §3 on what constitutes a Schubert calculus we must find algebraic generators of $H = S_W^{W_\theta}$ among the X_w, $w \in W^\theta$. Of course, in the case of S_W itself we had no problem, we simply chose the 1-dimensional classes X_s, $s \in S$. It is no longer true that H is generated by 1-dimensional classes. Fortunately, we can use the ideas of Chapter II to solve this problem. The map

$$S(V)^{W_\theta} \xrightarrow{c} H$$

is surjective. We also have

$$S(V)^{W_\theta} = \mathbb{C}[\tau_1,\ldots,\tau_k,\sigma_1,\ldots,\sigma_n]$$

where $\tau_i = s_i(e_1,\ldots,e_k)$, $1 \le i \le k$, and $\sigma_j = s_j(e_{k+1},\ldots,e_{k+n})$, $1 \le j \le n$, and s_j denotes the jth elementary symmetric function in an appropriate number of variables. The images $c(\sigma_j)$, $1 \le j \le n$ suffice to generate H (they are the special Schubert cycles of Chapter III, §3). We compute:

(5.4) *Lemma.* $c(\sigma_j) = (-1)^j X_{s_{k+j-1}\cdots s_k} = (-1)^j X(1,2,\ldots,k-1,k+j)$.

Proof. By (1.10*)

$$c(\sigma_j) = \sum_{\ell(w)=j} \Delta_w(\sigma_j) X_w$$

If we write Δ_t for Δ_{s_t}, then clearly $\Delta_t(\sigma_j) = 0$, if $t \neq k$ and

$$\Delta_k(\sigma_j) = \frac{s_j(e_{k+1},\ldots,e_{k+n}) - s_j(e_k,\ldots,e_{k+n})}{e_k - e_{k+1}}$$

$$= \frac{(e_{k+1}-e_k)s_{j-1}(e_{k+2},\ldots,e_{k+n})}{e_k - e_{k+k}}$$

$$= (-1)s_{j-1}(e_{k+2},\ldots,e_{k+n})$$

We can continue by induction and get $\Delta_{k+1-j}\cdots\Delta_k(\sigma_j) = (-1)^j$, while any other sequence of simple roots yields zero.

It remains to compute $X(1,2,\ldots,k-1,k+j)X(d_1,\ldots,d_k)$. The case $j = 1$ is easy.

(5.5) *Proposition.* In the algebra H

$$X(1,2,\ldots,k-1,k+1)X(d_1,\ldots,d_k) = \sum_{d_i+1<d_{i+1}} X(d_1,\ldots,d_i+1,\ldots,d_k)$$

Proof. Since $s_k = (1,2,\ldots,k-1,k+1)$, we can apply the case $i = k$ of (5.2) and observe $w(b) \ll w(t)$ if and only if $w(t) = w(b) + 1$.

To simplify notation we write X_i for X_{s_i}, $1 \le i < n+k$, and $X_{n+k} = 0$. We then have

(5.6) *Lemma.* $c(\sigma_j) = s_j(X_{k+1} - X_k, X_{k+2} - X_{k+1},\ldots,X_{n+k} - X_{n+k-1})$.

Proof. By the tables of [19], the ith fundamental weight is $\omega_i = e_1+\cdots+e_i - (\frac{i}{n+k})\sigma_1(e_1,\ldots,e_{n+k})$. Hence $\omega_i \equiv e_i+\cdots+e_i \pmod{I_W}$ and we get

$$c(\sigma_j) = c(s_j(e_{k+1},\ldots,e_{k+n}))$$

$$= c(s_j(\omega_{k+1}-\omega_k,\ldots,-\omega_{n+k-1}))$$

$$= s_j(X_{k+1}-X_k,\ldots,X_{n+k}-X_{n+k-1})$$

since c kills I_W and (2.1 ii).

This suggests the following computation.

(5.7) *Lemma.* For all i, $k+1 \le i \le k+n$, $w \in W$; in S_W

$$(X_i - X_{i-1})X_w = \sum_{\substack{i<t \\ w(i)<<w(t)}} X_{ws_{it}} - \sum_{\substack{k<b<i \\ w(b)<<w(i)}} X_{ws_{bi}} - \sum_{\substack{b\le k \\ w(b)<<w(i)}} X_{ws_{bi}}$$

Proof. Computing with (5.2), we get

$$X_{s_i}\cdot X_w = \sum_{\substack{b\le i-1 \\ i<t \\ w(b)<<w(t)}} X_{ws_{bt}} + \sum_{\substack{i<t \\ w(i)<<w(t)}} X_{ws_{it}}$$

and

$$X_{s_{i-1}}\cdot X_w = \sum_{\substack{b\le i-1 \\ i<t \\ w(b)<<w(t)}} X_{ws_{bt}} + \sum_{\substack{b<i \\ w(b)<<w(i)}} X_{ws_{bi}}$$

Upon subtracting and breaking up the second term the desired expression follows.

(5.8) *Theorem.* In the algebra H

$$s_j(X_{k+1}-X_k,\ldots,X_{n+k}-X_{n+k-1})\cdot X(d_1,\ldots,d_k) = (-1)^j\sum X(e_1,\ldots,e_k)$$

where the summation ranges over $(e_1,\ldots,e_k)$ satisfying $d_i \le e_i < d_{i+1}$ a
$\sum_{i=1}^{k} e_i = j + \sum_{i=1}^{k} d_i$.

Proof. We can write

$$s_j = \sum_{k+1\le t_1<\cdots<t_j\le k+n} (X_{t_j}-X_{t_j-1})\cdots(X_{t_1}-X_{t_1-1})$$

It is not difficult to check that the third term of (5.7) alone yield the right hand side of (5.8). Hence it remains to show that the contributions arising whenever either of the first two terms of (5.7) are involved cancel in the final summation. To do this it suffices to check that the resulting subscripts in W do not lie in W^θ. (Then they must have coefficient zero since H is a subalgebra of S_W).

Now the first two terms of (5.7) always give a transposition above $k+1$ and it must be an elementary one by (5.1), say s_i, $i \ge k$. Such a transposition will send an element of W^θ out of W^θ. We claim no further transposition s_{bt}, with either $b \ge i$ or $t \ge i$, will put the subscript back in W^θ. Both cases are easy to check and the proof is complete.

Finally by a substitution from (5.4) and multiplying all degrees by 2 w
get

(5.9) *Corollary*. (Classical Pieri Formula). In $H^*(G_k(\mathbb{C}^{n+k}))$

$$X(1,2,\ldots,k-1,k+j)X(d_1,\ldots,d_k) = \sum X(e_1,\ldots,e_k)$$

where the summation is as in (5.8).

The advantage of this approach to the Pieri formula is its suggestive gen
eralization to other G/P, P a maximal parabolic of G. Suppose G is a

group of type B_n and P_α is the maximal parabolic corresponding to omitting the "right-end" root $\alpha = \alpha_n$. (We return to this example in V, §3). This is the space SO_{2n+1}/U_n, the manifold of totally isotropic n-planes in a complex vector space of dimension $2n+1$ equipped with an appropriate orthogonal form. There is a map

$$S(V)^{\Sigma_n} \to H^*(G/P_\alpha)$$

It is also possible to compute this map explicitly, namely

$$c(\sigma_j) = 2X_{(j)}$$

where (j) denotes $s_{n+j-1}\cdots s_{n-1}s_n$ in $W(B_n)$. These Schubert varieties $X_{(j)}$, $1 \le j \le n$, play the role of the special Schubert cycles. The case $j = 1$ is worked out in Chapter V, §3. For $j > 1$, the result is complicated by multiplicities, but one can still follow the strategy used in this section. This result (and also the symplectic case) will be treated elsewhere [68].

§6. COMPLEMENTS

We tie up two loose ends here. The first is a geometric identification of our algebraic Schubert classes with the classes coming from the Bruhat decomposition. This result is due to Bernstein, Gelfand and Gelfand [6]. The second matter concerns an arithmetic property of the map $c:S(V) \to H^*(G/B)$. If we work integrally, c is not necessarily surjective. Can we describe the cokernel? Following Demazure [38], we see that the order of the cokernel gives the torsion primes of the Lie group G.

(6.1) It would be reassuring to have a result that guarantees that our algebraic basis of the coinvariant algebra S_W coincides with the geometr Schubert varieties of (III, §4). This amounts to checking $D_w = \varepsilon\Lambda_w$, $w \in$ where D_w is as defined at the beginning of §1. An argument for this app in Bernstein, *et. al.*[6] and we sketch an outline here.

Consider (3.4) above. If one could prove a similar formula with $\varepsilon\Lambda_w$ replaced by D_w, then we would be finished by induction on $\ell(w)$ [6 ,p. But we also have

$$\begin{aligned} D_w\omega_\alpha^*(f) &= D_w(\omega_\alpha f) \\ &= \langle X_w, c(\omega_\alpha f)\rangle \\ &= \langle X_w, X_{s_\alpha} \cup c(f)\rangle \\ &= \langle X_w \cdot X_{s_\alpha}, c(f)\rangle \end{aligned}$$

by (2.1 ii) and the fact that intersection $\cdot$ is adjoint to cup product. So it suffices to verify the following intersection formula for Schubert homology classes

$$X_w \cdot X_{s_\alpha} = \sum_{w' \overset{\alpha}{\to} w'} (w'^{-1}(\gamma), \omega_\alpha) X_{w'} \tag{5.1*}$$

as in (3.5). There is a fundamental representation V_α of G determined by ω_α which yields an embedding

$$G/B \to \mathbb{P}(V_\alpha)$$

where $\mathbb{P}$ denotes the projective space of lines. If we pullback the ampl line bundle on $\mathbb{P}(V_\alpha)$ we get a line bundle L_α on G/B. The element $w \in W$ yields a section on $\mathbb{P}(V_\alpha)$ which pulls back to a section ϕ_w of L_α. It turns out that computing the divisor of ϕ_w is equivalent to comp ting the coefficients in (5.1*). If $w' \overset{\alpha}{\to} w$ we get a map

$$j_\gamma : \mathbb{P}^1 = SL_2/B \to G/B$$

The multiplicity of χ_w, in $\chi_w \cdot \chi_{s_\alpha}$ is equal to the multiplicity of the zero of the function $j_\gamma^*(\phi_w)$ on $\mathbb{P}^1$. This is then computed using some elementary facts about representations of the Lie algebra $\mathfrak{sl}_2$ and the proof is finished (see also [77, Lemma 6.5] for this type of divisor computation).

(6.2) In (III, 5.3) we discussed briefly Demazure's analysis of the Chow ring of G/B [39]. This approach has the two-fold advantage of (a) replacing the complex manifold by a projective algebraic k-variety, k an arbitrary algebraically closed field and (b) working over the integers. Now the map c is not necessarily surjective, but Demazure proves that coker(c) is finite. If t is order of the cokernel of $c_N : S_N(V) \to (H_W)_N$ then t kills coker(c). The number t is called the *index of torsion* and its prime divisors are called the *torsion primes* of G. It is the smallest positive integer for which there exists a $u \in S_N$ with $J(u) = td$. Since $J(d) = |W|d$, $t \mid |W|$. Demazure's computation of the torsion primes agree with the more familiar result for the complex groups [13]. It is

G	torsion primes
A_n	1
B_n	2
C_n	1
D_n	2
E_6	2,3
E_7	2,3
E_8	2,3,5
F_4	2,3
G_2	2

In particular, if G is a product of special linear and symplectic groups, the map c is always surjective.

V Combinatorics of the Bruhat order

The Bruhat order on an arbitrary Coxeter group W was introduced in Chapter I, §6. More generally, one can consider the subset W^θ of coset representatives (see I, §5) with the order inherited from $W = W^\phi$.

Suppose now W is a finite Weyl group. Following [119], we refer to such posets (= partially ordered sets) W^θ as *Bruhat posets*. In Chapter III, §4 we gave a geometric interpretation of this poset in terms of the cell-decomposition of a certain homogeneous space G/P_θ. In particular, the Bruhat order on W describes the relative disposition of the Schubert varieties in a generalized flag manifold G/B. It is not unreasonable to expect that a better combinatorial understanding of the Bruhat poset W^θ will shed light on the geometry of these varieties G/P_θ. (Indeed, just such an application is worked out in §3). On the other hand, the Bruhat posets also provide an interesting and tractable class of examples for combinatorialists.

Here is a summary of this chapter. We begin in section 1 by collecting together some useful combinatorial jargon. It is intended to be a convenient reference for the other sections. Section 2 begins the study of intersection theory proper by identifying a reasonable class of parabolics P_θ to work with; the ones corresponding to a miniscule weight. The resulting varieties G/P_θ support noticeably simpler intersection theories.

We study an intersection problem in §3; namely take an arbitrary Schubert variety and successively intersect it with the unique codimension one subvariety until you are reduced to counting points. According to §2, the only interesting examples other than the Grassmann varieties are certain orthogo-