(i) Consider the two subalgebras (both of dimension 62):

$$N = \mathbb{K} \oplus \mathrm{Mat}_5(\mathbb{K}) \oplus \mathrm{Mat}_6(\mathbb{K})$$

$$\overline{N} = \mathrm{Mat}_2(\mathbb{K}) \oplus \mathrm{Mat}_3(\mathbb{K}) \oplus \mathrm{Mat}_7(\mathbb{K})$$

of the factor $M = \mathrm{Mat}_{12}(\mathbb{K})$, both inclusions being described by

$$(x,y,z) \mapsto \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}.$$

Then $\Lambda_N^M = \Lambda_{\overline{N}}^M = (1\ 1\ 1)$ though N and $\overline{N}$ are not isomorphic.

(ii) Consider $N = \mathbb{K} \oplus \mathrm{Mat}_2(\mathbb{K})$ included in $M = \mathrm{Mat}_4(\mathbb{K})$ by $(x,y) \mapsto \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}$ and in $\overline{M} = \mathrm{Mat}_5(\mathbb{K})$ by $(x,y) \mapsto \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}$. Then Λ_N^M and $\Lambda_N^{\overline{M}}$ are pseudo-equivalent to $(2\ 1)$ but M and $\overline{M}$ are not isomorphic.

(iii) Consider finally $N = \mathbb{K} \oplus \mathrm{Mat}_2(\mathbb{K})$ included in $M = \mathrm{Mat}_5(\mathbb{K})$ by $(x,y) \mapsto \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{bmatrix}$ and by $(x,y) \mapsto \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}$. Then the first inclusion matrix $(3\ 1)$ is not pseudo-equivalent to the second inclusion matrix $(1\ 2)$.

The next proposition is a special case of a statement which appears in [BA 8], §5, exercise 17.

Proposition 2.3.5. *Consider two multi-matrix subalgebras* M,N *of a factor* F *with* $1 \in N \subset M \subset F$. *The inclusion matrix for* $C_F(M) \subset C_F(N)$ *is the transpose of the inclusion matrix for* $N \subset M$.

Proof. The proposition is obvious if M and N are factors (see the Remark following 2.2.2). In general, write

$$M = \bigoplus_{i=1}^{m} p_i M \qquad N = \bigoplus_{j=1}^{n} q_j N \qquad \Lambda_N^M = (\lambda_{i,j})$$

and denote by $\tilde{\lambda}_{j,i}$ the entries of the inclusion matrix for

$$C_F(N) = \bigoplus_{j=1}^{n} q_j C_F(N) \supset C_F(M) = \bigoplus_{i=1}^{n} p_i C_F(M).$$

One has by definition

$$\tilde{\lambda}_{j,i} = [q_j p_i C_F(N) q_j p_i : q_j p_i C_F(M) q_j p_i]^{1/2}$$

and by Proposition 2.2.5.b,

$$\tilde{\lambda}_{j,i} = [C_{q_j p_i F q_j p_i}(N_{i,j}) : C_{q_j p_i F q_j p_i}(M_{i,j})]^{1/2}.$$

As $N_{i,j}$ and $M_{i,j}$ are factors in $q_j p_i F q_j p_i$ one has

$$\tilde{\lambda}_{j,i} = [M_{i,j} : N_{i,j}]^{1/2}$$

by the particular case observed in the remark following 2.2.2. #

The Bratteli diagram.

It is useful to describe a pair of multi-matrix algebras $N \subset M$ by its Bratteli diagram $B(N \subset M)$, which is a bicolored weighted multigraph defined as follows. ("Multigraph" means that two points may be joined by more than one line, "weighted" means that each point is given together with a positive integer, and "bicolored" means that points are given one of two colors, in such a way that any edge in the multigraph connects points of different colors.) If $M = \bigoplus_{i=1}^{m} \mathrm{Mat}_{\mu_i}(\mathbb{K})$ and $N = \bigoplus_{j=1}^{n} \mathrm{Mat}_{\nu_j}(\mathbb{K})$ are as above, then $B(N \subset M)$ has m black vertices $b_1,\cdots,b_m$ with respective weights $\mu_1,\cdots,\mu_m$ and n white vertices $w_1,\cdots,w_n$ with respective weights $\nu_1,\cdots,\nu_n$; moreover, the $i^{\underline{th}}$ black vertex and the $j^{\underline{th}}$ white vertex are joined by $\lambda_{i,j}$ lines. (These diagrams were first introduced in order to study inductive limit systems of finite dimensional C^*-algebras; see [Bra] and [Eff].)

Example 2.3.6. If $N = \mathrm{Mat}_\nu(\mathbb{C}) \otimes 1 \subset M = \mathrm{Mat}_\nu(\mathbb{C}) \otimes \mathrm{Mat}_3(\mathbb{C})$, then $B(N \subset M)$ is 

and $\Lambda_N^M = [3]$.

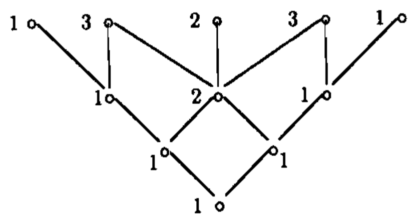
Example 2.3.7. Let G be a finite group and let H be a subgroup of G. As complex group algebras are semi-simple by Maschke's theorem (example II.2), $\mathbb{C}[H] \subset \mathbb{C}[G]$ is a multi-matrix algebra pair.

In particular, let $\mathfrak{S}_3$ be the group of permutations of $\{1,2,3\}$ and let $\mathfrak{S}_2$ be that of $\{1,2\}$. Minimal central idempotents of $\mathbb{C}[\mathfrak{S}_3]$ correspond to Young frames, and also to

Chains of multi-matrix algebras. Now consider an increasing chain (finite or infinite) of multi-matrix algebras over $\mathbb{K}$,

$$\mathbf{1} \in M_0 \subset M_1 \subset M_2 \subset \cdots .$$

Let $p_1^k, \cdots, p_{m(k)}^k$ denote the minimal central idempotents in M_k, let $\Lambda^{(k)} = (\lambda_{i,j}^k)$ be the inclusion matrix for $M_k \subset M_{k+1}$, and let μ^k be the vector of dimensions of M_k, so that $p_i^k M_k \cong \mathrm{Mat}_{\mu_i^k}(\mathbb{K})$. (Thus $\mu^k = \Lambda^{(k-1)}\Lambda^{(k-2)} \cdots \Lambda^{(0)}\mu^0$.) We associate with the chain of algebras a (finite or infinite) Bratteli diagram B, which is the union of the diagrams $B(M_k \subset M_{k+1})$, the upper (black) vertex of $B(M_k \subset M_{k+1})$ corresponding to p_i^{k+1} being identified with the lower (white) vertex of $B(M_{k+1} \subset M_{k+2})$ corresponding to the same idempotent. For example the diagram for $\mathbb{C}\mathfrak{S}_1 \subset \mathbb{C}\mathfrak{S}_2 \subset \mathbb{C}\mathfrak{S}_3 \subset \mathbb{C}\mathfrak{S}_4$ is



(See examples 2.3.7 and 2.3.8.) We say that the vertices v_i^k corresponding to the minimal central idempotents p_i^k in M_k belong to the k^{th} floor of the diagram. The vertices of the k^{th} and $k+1^{st}$ floors together with the edges joining them — i.e., the image of $B(M_k \subset M_{k+1})$ in B — constitute the k^{th} story of B. The Bratelli diagram B is thus a weighted multigraph with the following features:

(1) There is a function φ from the set of vertices of B to $\mathbb{N} = \{0,1,2,\cdots\}$, which assigns to each vertex the floor which it occupies.

(a) There are only finitely many vertices on each floor; that is $\varphi^{-1}(k)$ is finite for all k. If $\varphi^{-1}(k) \neq \emptyset$, we write $\varphi^{-1}(k) = \{v_1^k, \cdots, v_{m(k)}^k\}$.

(b) The range of φ is either an interval $[0,\rho]$ in $\mathbb{N}$, if B is finite, or all of $\mathbb{N}$, if B is infinite.

(2) Two vertices v and w are adjacent only if $|\varphi(v) - \varphi(w)| = 1$. There are $\lambda_{i,j}^k$ edges joining v_j^k and v_i^{k+1}.

(3) If both the k^{th} and $k+1^{st}$ floors are occupied (i.e., if $\varphi^{-1}(k) \neq \emptyset$ and $\varphi^{-1}(k+1) \neq \emptyset$) then each vertex on the k^{th} floor is adjacent to at least one vertex on the $k+1^{st}$ floor, and each vertex on the $k+1^{st}$ floor is adjacent to at least one vertex

on the k^{th} floor. That is, the $m(k)$-by-$m(k+1)$ matrix $\Lambda^{(k)} = (\lambda_{i,j}^k)$ is irredundant.

(4) Each vector v_i^k has a weight $\mu_i^k \in \{1,2,\cdots\}$ called its dimension. The dimensions $\{\mu_i^k\}$ and the "multiplicities" $\{\lambda_{i,j}^k\}$ satisfy

$$\sum_{j=1}^{m(k)} \lambda_{i,j}^k \mu_j^k = \mu_i^{k+1}.$$

Conversely, given a weighted multigraph B with properties (1)-(4) above, we can, by iterating the procedure of Proposition 2.3.9.b, construct a chain of multi-matrix algebras with Bratteli diagram B.

Proposition 2.3.10. *Suppose*

$$\mathbf{1} \in M_0 \subset M_1 \subset \cdots, \text{ and}$$
$$\mathbf{1} \in A_0 \subset A_1 \subset \cdots$$

are two chains of multi-matrix algebras with the same Bratteli diagram. Then there is an isomorphism ψ *of* $M_\infty = \bigcup_k M_k$ *onto* $A_\infty = \bigcup_k A_k$ *such that* $\psi(M_k) = A_k$ *for all* k.

Proof. We have to produce a sequence of isomorphisms $\psi_k : M_k \to A_k$ such that $\psi_{k+1}\big|_{M_k} = \psi_k$. Let $\psi_0 : M_0 \to A_0$ be any isomorphism. Suppose $\psi_0, \cdots, \psi_k$ have been defined. Then by Proposition 2.3.9.a, there is an isomorphism $\alpha_{k+1} : M_{k+1} \to A_{k+1}$ such that $\alpha_{k+1}(M_k) = A_k$, and by Proposition 2.3.3 there is an inner automorphism β_{k+1} of A_{k+1} extending $\psi_k \circ \alpha_{k+1}^{-1}\big|_{A_k}$. Thus we can set $\psi_{k+1} = \beta_{k+1} \circ \alpha_{k+1}$. #

2.3.11. A path model. Let B be a Bratteli diagram; we use paths on the diagram to construct a natural model for the chain of multi-matrix algebras associated to the diagram. We will suppose that B is infinite; it will be obvious how the construction must be modified for a finite diagram. First we produce an augmented diagram $\tilde{B}$ by adding a $(-1)^{st}$ story corresponding to the inclusion $\mathbb{K}\mathbf{1} \subset M_0$; that is we append one vertex $*$ with $\varphi(*) = -1$ and $\dim(*) = 1$, and we connect $*$ to v_j^0 by μ_j^0 edges $(1 \le j \le m(0))$.

An oriented edge on any graph is an edge together with an ordering of its two vertices; we will call the first vertex of an oriented edge its start and the second its end. A path is a (possibly infinite) sequence (ξ_i) of oriented edges such that $\mathrm{end}(\xi_i) = \mathrm{start}(\xi_{i+1})$ for all i. A path $(\cdots,\xi_k)$ has end equal to $\mathrm{end}(\xi_k)$; a path $(\xi_0,\cdots)$ has start equal to $\mathrm{start}(\xi_0)$. If ξ and η are paths such that $\mathrm{end}(\xi) = \mathrm{start}(\eta)$ we define $\xi\circ\eta$ to be the path "first ξ, then η". A path ξ on $\tilde{B}$ is <u>monotone increasing</u> if $\varphi(\mathrm{end}(\xi_k)) = \varphi(\mathrm{start}(\xi_k)) + 1$ for all k.

We let Ω denote the set of infinite monotone increasing paths on $\tilde{B}$ starting at $*$; $\Omega_{[r}$ the set of infinite monotone increasing paths starting on the r^{th} floor of $\tilde{B}$; $\Omega_{r]}$ the set of monotone increasing paths starting at $*$ and ending on the r^{th} floor; and $\Omega_{[r,s]}$ the set of monotone increasing paths starting on the r^{th} floor and ending on the s^{th} floor $(r < s)$. Given $\xi = (\xi_0,\xi_1,\cdots) \in \Omega$, set:

$$\begin{aligned} \xi_{r]} &= (\xi_0,\cdots,\xi_r) \in \Omega_{r]} && (0 \le r),\\ \xi_{[r,s]} &= (\xi_{r+1},\cdots,\xi_s) \in \Omega_{[r,s]} && (-1 \le r < s),\\ \text{and}\quad \xi_{[r} &= (\xi_{r+1},\cdots) \in \Omega_{[r} && (-1 \le r).\end{aligned}$$

Also let $\xi_{[r]}$ be the vertex $\mathrm{end}(\xi_r) = \mathrm{start}(\xi_{r+1})$. Similarly if $\xi = (\xi_0,\cdots\xi_s) \in \Omega_{s]}$ and $r \le s$ we can define $\xi_{r]} = (\xi_0,\cdots,\xi_r) \in \Omega_{r]}$, and so forth.

Let $\mathbb{K}\Omega$ be the $\mathbb{K}$-vector space with basis Ω. For each $r \in \{0,1,2,\cdots\}$ we define an algebra $A_r \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega)$ as follows. Let $R_r = \{(\xi,\eta) \in \Omega_{r]} \times \Omega_{r]} : \mathrm{end}(\xi) = \mathrm{end}(\eta)\}$. For $(\xi,\eta) \in R_r$ define $T_{\xi,\eta} \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega)$ by

$$T_{\xi,\eta}\omega = \delta(\eta_{r]},\omega_{r]})\xi_{r]}\circ\omega_{[r} \quad (\omega \in \Omega).$$

Let A_r be the $\mathbb{K}$-linear span of $\{T_{\xi,\eta} : (\xi,\eta) \in R_r\}$ in $\mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega)$; since

(2.3.11.1) $$T_{\xi,\eta}T_{\xi',\eta'} = \delta(\eta,\xi')T_{\xi,\eta'}, \text{ and } \mathbf{1} = \sum_{\xi\in\Omega_{r]}} T_{\xi,\xi}$$

A_r is an algebra. Set

$$\Omega^i_{r]} = \{\xi \in \Omega_{r]} : \mathrm{end}(\xi) = v^r_i\} \quad (1 \le i \le m(r)),$$

so that $\Omega_{r]} = \biguplus_i \Omega^i_{r]}$ (disjoint union), and $R_r = \biguplus_i (\Omega^i_{r]} \times \Omega^i_{r]})$. It follows from the multiplication law (2.3.11.1) for the $T_{\xi,\eta}$ that

$$A^i_r = \mathrm{span}\{T_{\xi,\eta} : (\xi,\eta) \in \Omega^i_{r]}\times\Omega^i_{r]}\}$$

is an ideal of A_r and $A_r = \bigoplus_{i=1}^{m(r)} A^i_r$. There is an isomorphism of A^i_r onto $\mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega^i_{r]})$ defined by

$$T_{\xi,\eta}\omega = \delta(\eta,\omega)\xi \qquad (\xi,\eta,\omega \in \Omega^i_{r]}),$$

so that

$$A_r = \bigoplus_{i=1}^{m(r)} A^i_r \cong \bigoplus_{i=1}^{m(r)} \mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega^i_{r]}).$$

Note that the minimal central projections p^r_i in A_r have the form

$$p^r_i = \sum\{T_{\xi,\xi} : \xi \in \Omega^i_{r]}\}.$$

The cardinalities $\#(\Omega^i_{r]})$ satisfy

$$\#(\Omega^i_{0]}) = \mu^0_i, \text{ and } \#(\Omega^i_{r+1]}) = \sum_{j=1}^{m(r)} \lambda^r_{i,j}\,\#(\Omega^j_{r]}),$$

since each $\xi \in \Omega^j_{r]}$ can be extended in $\lambda^r_{i,j}$ ways, by adjunction of an edge λ in $\Omega_{[r,r+1]}$, to a path $\xi\circ\lambda$ in $\Omega^i_{r+1]}$. It follows from this and property (4) of the Bratteli diagram that $\#(\Omega^i_{r]}) = \mu^r_i$ for all r and i $(0 \le r, 1 \le i \le m(r))$. Thus

$$A_r \cong \bigoplus_{i=1}^{m(r)} \mathrm{Mat}_{\mu^r_i}(\mathbb{K}).$$

Finally $A_r \subset A_{r+1}$, because for $(\xi,\eta) \in R_r$,

$$T_{\xi,\eta} = \sum\{T_{\xi\circ\lambda,\eta\circ\lambda} : \lambda \in \Omega_{[r,r+1]}, \mathrm{end}(\xi) = \mathrm{start}(\lambda)\},$$

as operators on $\mathbb{K}\Omega$. If $(\xi,\eta) \in \Omega^j_{r]} \times \Omega^j_{r]}$, so $T_{\xi,\eta} \in A^j_r$, then

$$T_{\xi,\eta}p^{r+1}_i = \sum\{T_{\xi\circ\lambda,\eta\circ\lambda} : \lambda \in \Omega_{[r,r+1]}, \lambda_{[r]} = v^r_j, \lambda_{[r+1]} = v^{r+1}_i\}.$$

It follows that $\Lambda_{A_r}^{A_{r+1}} = (\lambda^r_{i,j})$, and the Bratteli diagram for the chain $\mathbf{1} \in A_0 \subset A_1 \subset \cdots$ is B.

As an example of the utility of the path model, let us identify $C_{A_s}(A_r)$ for $r < s$. Let

$$R_{r,s} = \{(\xi,\eta) \in \Omega_{[r,s]} \times \Omega_{[r,s]} : \xi_{[r]} = \eta_{[r]} \text{ and } \xi_{[s]} = \eta_{[s]}\}.$$

For $(\xi,\eta) \in R_{[r,s]}$ define $T_{\xi,\eta} \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}\Omega)$ by

$$T_{\xi,\eta}\omega = \delta(\eta_{[r,s]},\omega_{[r,s]})\omega_{r]}\circ\xi_{[r,s]}\circ\omega_{[s},$$

and let $A_{r,s} = \mathrm{span}_{\mathbb{K}}\{T_{\xi,\eta} : (\xi,\eta) \in R_{r,s}\}$. Then $A_{r,s}$ is an algebra, since again

$$T_{\xi,\eta}T_{\xi',\eta'} = \delta(\eta,\xi')T_{\xi,\eta'} \quad ((\xi,\eta),(\xi',\eta') \in R_{r,s}), \text{ and } \mathbf{1} = \sum\{T_{\xi,\xi} : \xi \in \Omega_{[r,s]}\}.$$

We have $A_{r,s} \subset A_s$, because if $(\xi,\eta) \in R_{r,s}$, then

$$T_{\xi,\eta} = \sum\{T_{\lambda\circ\xi,\lambda\circ\eta} : \lambda \in \Omega_{r]}, \lambda_{[r]} = \xi_{[r]} = \eta_{[r]}\},$$

as operators on $\mathbb{K}\Omega$. Clearly $A_{r,s} \subset C_{A_s}(A_r)$.

<u>Proposition 2.3.12.</u> $A_{r,s} = C_{A_s}(A_r)$.

<u>Proof.</u> For $x \in A_s$ define $P(x) =$

$$\sum\{T_{\lambda,\lambda'}xT_{\lambda',\lambda} : (\lambda,\lambda') \in R_r\}.$$

One verifies that P is a linear projection of A_s onto $C_{A_s}(A_r)$. But for $(\xi,\eta) \in R_s$,

$$\begin{aligned} P(T_{\xi,\eta}) &= \delta(\xi_{r]},\eta_{r]}) \sum\{T_{\lambda\circ\xi_{[r},\lambda\circ\eta_{[r}} : \lambda \in \Omega_{r]}, \mathrm{end}(\lambda) = \xi_{[r]}\} \\ &= \delta(\xi_{r]},\eta_{r]})T_{\xi_{[r},\eta_{[r}} \in A_{r,s}. \end{aligned}$$

Thus $C_{A_s}(A_r) \subset A_{r,s}$. #

It is an easy exercise to check that the factors of $A_{r,s}$ are in bijection with pairs of vertices (v,w), with v in floor r and w in floor s. The factor corresponding to a pair (v,w) is the algebra of endomorphisms of the free vector space over the set of paths from v to w.

<u>Remarks.</u> (1) The path model presented here is due to V.S. Sunder [Sun] and A. Ocneanu [Ocn]. Compare however [SV], in which a maximal abelian subalgebra of $A_\infty = \bigcup_k A_k$ is identified with $\mathbb{K}\Omega$.

(2) In case $\mathbb{K} = \mathbb{C}$, the action of the "path algebras" A_r on $\mathbb{C}\Omega$ extends to a representation on the Hilbert space $\ell^2(\Omega)$ with orthonormal basis Ω. It is evident that $T_{\xi,\eta}$ is then a rank-one partial isometry with adjoint $T^*_{\xi,\eta} = T_{\eta,\xi}$. So A_r is a C^*-subalgebra of $\mathcal{B}(\ell^2(\Omega))$.

2.4. The fundamental construction and towers for multi-matrix algebras.

We consider a pair of multi-matrix algebras $\mathbf{1} \in N \subset M$, and the associated tower of algebras

$$\mathbf{1} \in M_0 = N \subset M_1 = M \subset \cdots \subset M_k \subset M_{k+1} \subset \cdots$$

obtained by iterating the fundamental construction, as described in the chapter introduction. It turns out that all the M_k are then multi-matrix algebras:

<u>Proposition 2.4.1.</u> *Let* $N \subset M$ *be a pair of multi-matrix algebras and let* $M \subset \mathrm{End}^r_N(M)$ *be the pair obtained by the fundamental construction. Then*

(a) $\mathrm{End}^r_N(M)$ *is a multi-matrix algebra and its minimal central idempotents are of the form* $\rho(q)$, *where* q *is a minimal central idempotent in* N, *and* $\rho(q)$ *is <u>right</u> multiplication by* q.

(b) *The inclusion matrix for* $M \subset \mathrm{End}^r_N(M)$ *is the transpose of* Λ^M_N.

<u>Proof.</u> Set $F = \mathrm{End}_{\mathbb{K}}(M)$ and define maps $\lambda,\rho : M \longrightarrow F$ by $\lambda(x)(y) = xy$ and $\rho(x)(y) = yx$ for $x,y \in M$. The homomorphism λ is the composition of the inclusions $M \subset \mathrm{End}^r_N(M)$ and $\mathrm{End}^r_N(M) \subset F$; the map ρ is an algebra isomorphism from M^{opp} into F. As the pair $N \subset M$ is isomorphic to the pair $N^{\mathrm{opp}} \subset M^{\mathrm{opp}}$ by Corollary 2.3.4, it is also isomorphic to $\rho(N) \subset \rho(M)$. But $\mathrm{End}^r_N(M) = C_F(\rho(N))$ and $M = \lambda(M) = C_F(\rho(M))$. Consequently (a) follows from 2.2.3.a and (b) from 2.3.5. #

irreducible representations of $\mathfrak{S}_3$. We denote them by

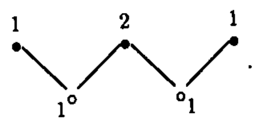
$p_{(3)}$ corresponding to the identity representation $\pi_{(3)}$ of $\mathfrak{S}_3$
$p_{(1,1,1)}$ corresponding to the signature representation $\pi_{(1,1,1)}$ of $\mathfrak{S}_3$
$p_{(2,1)}$ corresonding to the 2-dimensional irreducible representation $\pi_{(2,1)}$ of $\mathfrak{S}_3$.

Similarly for

$q_{(2)}$ corresponding to the identity representation $\pi_{(2)}$ of $\mathfrak{S}_2$
$q_{(1,1)}$ corresponding to the signature representation $\pi_{(1,1)}$ of $\mathfrak{S}_2$.

It is easy to check that the representations $\pi_{(3)}$, $\pi_{(2,1)}$, $\pi_{(1,1,1)}$ of $\mathfrak{S}_3$ restrict to $\mathfrak{S}_2$ respectively as $\pi_{(2)}$, $\pi_{(2)} \oplus \pi_{(1,1)}$, $\pi_{(1,1)}$. It follows that the inclusion matrix and the Bratteli diagram for $\mathbb{C}[\mathfrak{S}_2] \subset \mathbb{C}[\mathfrak{S}_3]$ are

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$



Example 2.3.8. Consider similarly $\mathfrak{S}_3$ as a subgroup of the group $\mathfrak{S}_4$ of permutations of $\{1,2,3,4\}$. The group $\mathfrak{S}_4$ has irreducible representations

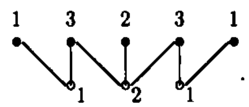
$$\pi_{(4)} \quad \pi_{(3,1)} \quad \pi_{(2,2)} \quad \pi_{(2,1,1)} \quad \pi_{(1,1,1,1)}$$

of respective dimensions 1,3,2,3,1, whose restrictions to $\mathfrak{S}_3$ are respectively

$$\pi_{(3)} \quad \pi_{(3)} \oplus \pi_{(2,1)} \quad \pi_{(2,1)} \quad \pi_{(2,1)} \oplus \pi_{(1,1,1)} \quad \pi_{(1,1,1)};$$

see, for example, [Ser1], Example 5.8. It follows that the inclusion matrix and the Bratteli diagram for $\mathbb{C}[\mathfrak{S}_3] \subset \mathbb{C}[\mathfrak{S}_4]$ are as follows. (The reader will check that $\Lambda\vec{\nu} = \vec{\mu}$.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



As in the examples, we always draw Bratteli diagrams on two levels, with the upper level representing the larger algebra; the coloring of the vertices is actually superfluous, since the two types of vertices are labelled by their level. The equation $\Lambda\vec{\nu} = \vec{\mu}$ has the following interpretation: For a given black vertex v, consider the set of edges entering v, and for each edge take the weight of the white vertex incident to that edge. The sum of these weights, over all such edges, is the weight of v.

Proposition 2.3.9. (a) *Let* $N \subset M$ *and* $\overline{N} \subset \overline{M}$ *be two multi-matrix algebra pairs with the same Bratteli diagram. Then there exists an isomorphism* $\theta : M \longrightarrow \overline{M}$ *with* $\theta(N) = \overline{N}$.

(b) *A bicolored weighted multigraph* B *(with positive integer weights) is the Bratteli diagram of a multi-matrix algebra pair if and only if the weights and the multiplicities* $\lambda_{i,j}$ *satisfy* $\mu_i = \sum_j \lambda_{i,j}\nu_j$.

Proof. As (a) is nothing but a restatement of Proposition 2.3.3, we are left with the proof of (b).

Let $\mu_1,\cdots,\mu_m$ be the weights of the black points in B and let $\nu_1,\cdots,\nu_n$ be those of the white points and suppose $\mu_i = \sum_j \lambda_{i,j}\nu_j$. Set

$$M = \bigoplus_{j=1}^{m} \mathrm{Mat}_{\mu_i}(\mathbb{K}) \qquad N = \bigoplus_{j=1}^{n} \mathrm{Mat}_{\nu_j}(\mathbb{K}).$$

Let $\lambda_{i,j}$ be the number of lines joining the $i^{\underline{th}}$ black point with the $j^{\underline{th}}$ white point in B. Define a map $N \longrightarrow M$ by associating to $(y_1,\cdots,y_n) \in N$ the element $(x_1,\cdots,x_m) \in M$ with x_i the block-diagonal matrix

$$x_i = \mathrm{diag}(y_1,\cdots,y_1;\cdots;y_n,\cdots,y_n)$$

where y_j is repeated $\lambda_{i,j}$ times. This map identifies N with a subalgebra of M and $B(N \subset M)$ is the B originally given. #